Outline

- 9.1 Graph and Graph Models
- 9.2 Graph Terminology and Special Types of Graphs
- 9.3 Representing Graphs and Graph Isomorphism
- 9.4 Connectivity
- 9.5 Euler and Hamiltonian Paths
- 9.6 Shortest-Path Problems
- 9.7 Planar Graphs
- 9.8 Graph Coloring

Outline

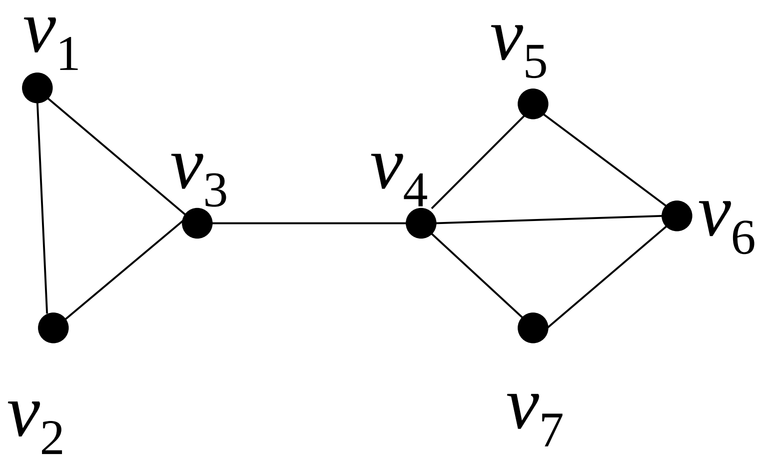
- 9.1 Graph and Graph Models
- 9.2 Graph Terminology and Special Types of Graphs
- 9.3 Representing Graphs and Graph Isomorphism
- 9.4 Connectivity
- 9.5 Euler and Hamiltonian Paths
- 9.6 Shortest-Path Problems
- 9.7 Planar Graphs
- 9.8 Graph Coloring

§9.1 Introduction to Graphs

A **graph** is a **mathematical** way of representing the concept of a "network“.

Def 1. A graph $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ consists of $\boldsymbol{V}$, a nonempty set of vertices (or nodes), and $\boldsymbol{E}$, a set of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints.

eg.



$G=(V, E)$, where

$V=\{v_1,v_2,\ldots, v_7\}$

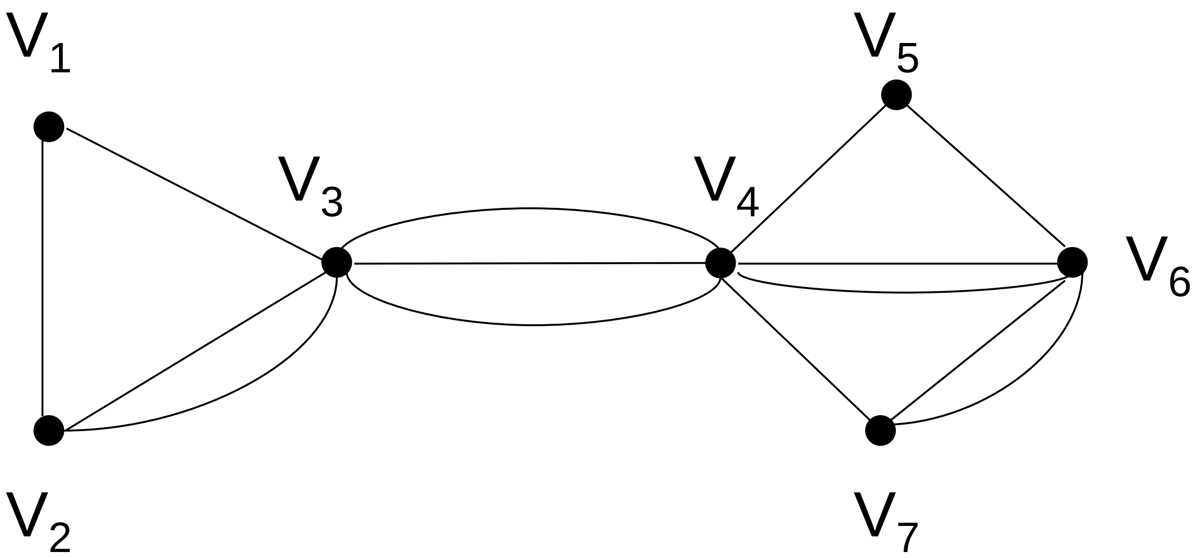
$E=\{\{v_1,v_2\}, \{v_1,v_3\}, \{v_2,v_3\}$
$\{v_3,v_4\}, \{v_4,v_5\}, \{v_4,v_6\}$
$\{v_4,v_7\}, \{v_5,v_6\}, \{v_6,v_7\}\}$

Def A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a simple graph.

Def Multigraph:

simple graph + multiple edges (multiedges)
(兩點間允許多條邊)

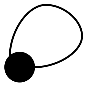
eg.



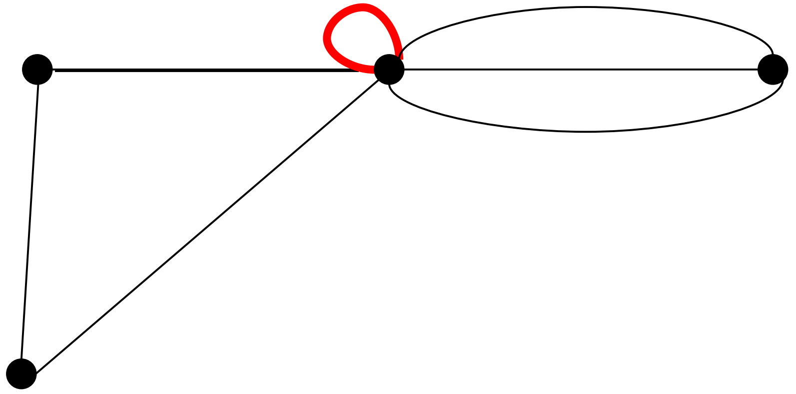
Def. Pseudograph:

simple graph + multiedge

\+ loop

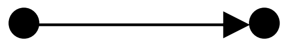
(a loop: )

eg.



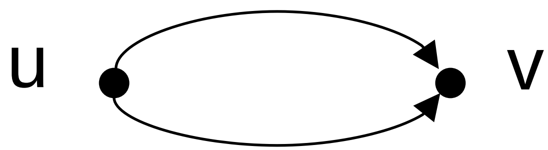
Def 2. Directed graph (digraph):

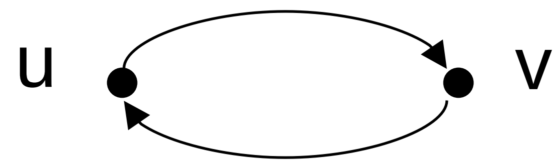
simple graph with each edge directed



Note:  is allowed in a directed graph

Note:





The two edges (u,v),(u,v) are multiedges.

The two edges (u,v), (v,u) are not multiedges.

Def. Directed multigraph: digraph+multiedges

Table 1. Graph Terminology

Type	Edges	Multiple Edges	Loops
(simple) graph	undirected edge: {u,v}	✗	✗
Multigraph		✓	✗
Pseudograph		✓	✓
Directed graph	directed edge: (u,v)	✗	✓
Directed multigraph		✓	✓

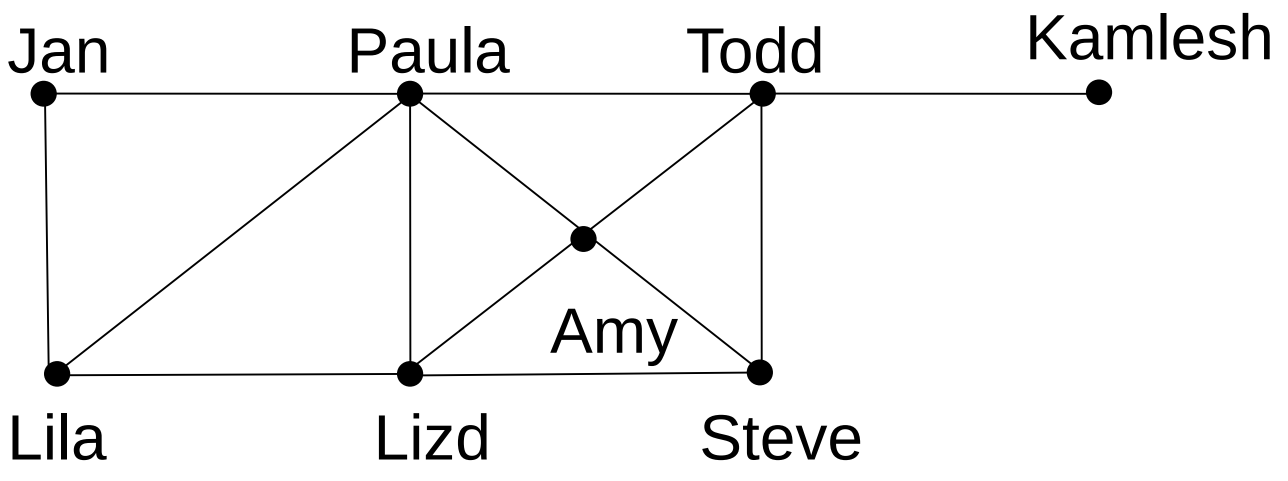
Exercise : 3, 5, 6, 7, 9

Graph Models

Example 2. (Acquaintanceship graphs)

We can use a simple graph to represent whether two people know each other. Each person is represented by a vertex. An undirected edge is used to connect two people when these people know each other.

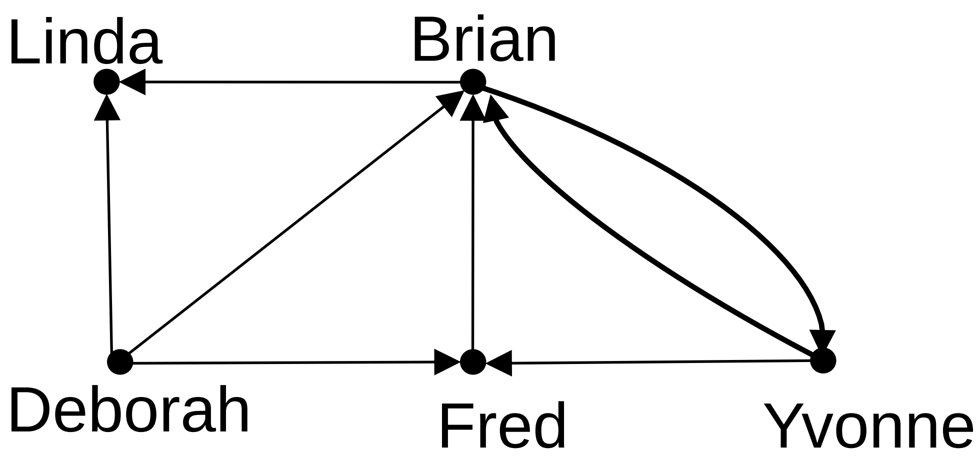
eg



Example 3. (Influence graphs)

In studies of group behavior it is observed that certain people can influence the thinking of others. Simple digraph $\Rightarrow$ Each person of the group is represented by a vertex. There is a directed edge from vertex a to vertex b when the person a influences the person b.

eg



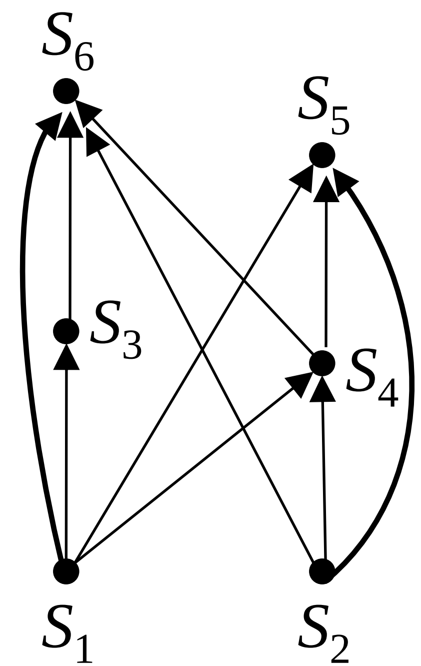
Example 9. (Precedence graphs and concurrent processing)

Computer programs can be executed more rapidly by executing certain statements concurrently. It is important not to execute a statement that requires results of statements not yet executed.

Simple digraph $\Rightarrow$ Each statement is represented by a vertex, and there is an edge from a to b if the statement of b cannot be executed before the statement of a.

eg

S_1: a:=0
S_2: b:=1
S_3: c:=a+1
S_4: d:=b+a
S_5: e:=d+1
S_6: e:=c+d



Ex 13. The intersection graph of a collection of sets $A_1, A_2, \ldots, A_n$ is the graph that has a vertex for each of these sets and has an edge connecting the vertices representing two sets if these sets have a nonempty intersection. Construct the intersection graph of the following collection of sets.

(a) $A_1 = \{0, 2, 4, 6, 8\}$, $A_2 = \{0, 1, 2, 3, 4\}$,
$A_3 = \{1, 3, 5, 7, 9\}$, $A_4 = \{5, 6, 7, 8, 9\}$, $A_5 = \{0, 1, 8, 9\}$.

(請做此作業)

§9.2 Graph Terminology

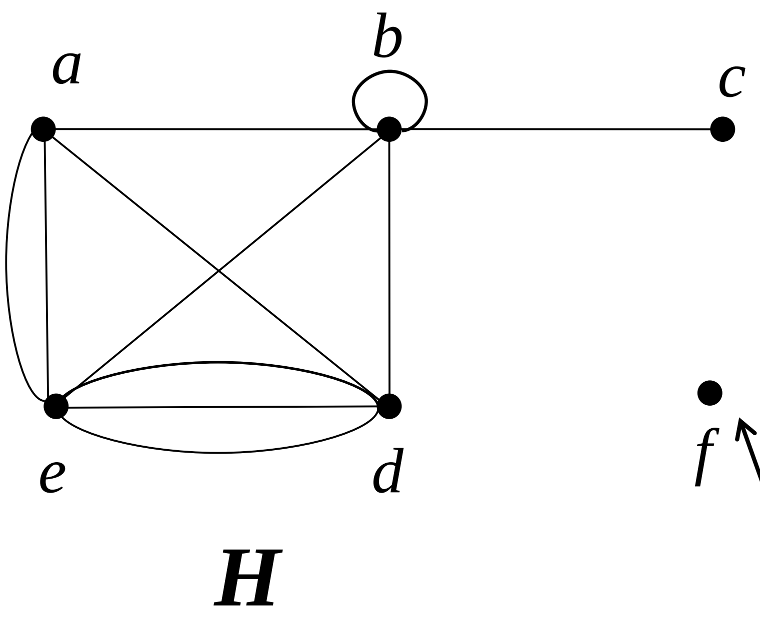
Def 1. Two vertices u and v in a undirected graph G are called adjacent (or neighbors) in G if $\{u, v\}$ is an edge of G.

Note : adjacent: a vertex connected to a vertex
incident: a vertex connected to an edge

Def 2. The degree of a vertex v, denoted by deg(v), in an undirected graph is the number of edges incident with it.

(Note : A loop adds 2 to the degree.)

Example 1. What are the degrees of the vertices in the graph H ?



Sol :

$\deg(a)=4$

$\deg(b)=6$

$\deg(c)=1$

$\deg(d)=5$

$\deg(e)=6$

$\deg(f)=0$

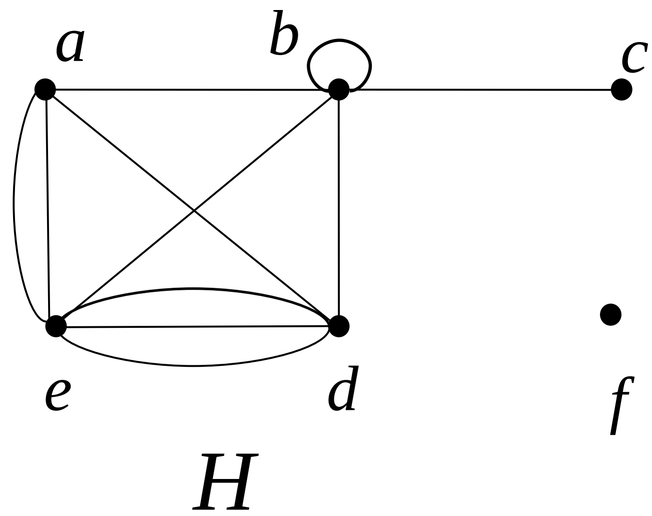
Def. A vertex of degree 0 is called isolated.

Thm 1. (The Handshaking Theorem)

Let $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ be an undirected graph with e edges (i.e., $|\boldsymbol{E}| = \boldsymbol{e}$). Then

$$\sum_{v \in V} \deg(v) = 2e$$

Pf : 每加一條 edge $\{u, v\}$ 會同時使 u 跟 v 各增加一個degree eg.



The graph H has 11 edges, and

$$\sum_{v \in V} \deg(v) = 22$$

Example 3. How many edges are there in a graph with 10 vertices each of degree six?

Sol :

$$10 \cdot 6 = 2e \quad \Rightarrow \quad e = 30$$

Thm 2. An undirected graph $G = (V, E)$ has an even number of vertices of odd degree.

Pf : Let $V_1=\{v\in V \mid \deg(v) \text{ is even}\}$,

$V_2=\{v\in V \mid \deg(v) \text{ is odd}\}$.

$$2e = \sum_{v\in V_1} \deg(v) + \sum_{v\in V_2} \deg(v) \quad \Rightarrow \quad \sum_{v\in V_2} \deg(v) \text{ is even.}$$

Exercise : 5

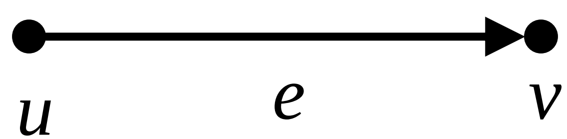
Def 3. $G = (V, E)$: directed graph,

$e = (u, v) \in E$: u is adjacent to v

v is adjacent from u

u : initial vertex of e

v : terminal (end) vertex of e



Def 4.

$G = (V, E)$: directed graph, $v \in V$

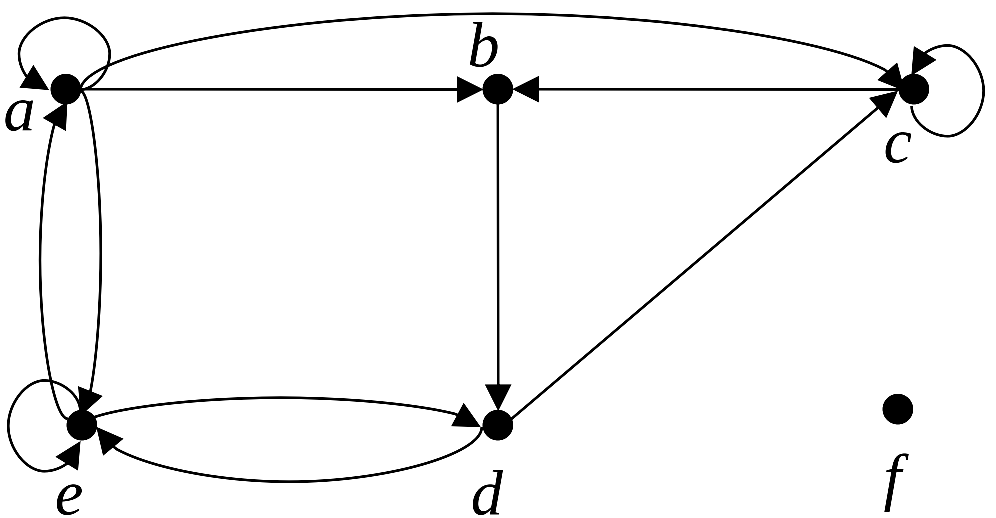
$\deg^{-}(v)$: # of edges with v as a terminal.

(in-degree)

$\deg^{+}(v)$: # of edges with v as a initial vertex

(out-degree)

Example 4.



$\deg^{-}(a)=2, \deg^{+}(a)=4$

$\deg^{-}(b)=2, \deg^{+}(b)=1$

$\deg^{-}(c)=3, \deg^{+}(c)=2$

$\deg^{-}(d)=2, \deg^{+}(d)=2$

$\deg^{-}(e)=3, \deg^{+}(e)=3$

$\deg^{-}(f)=0, \deg^{+}(f)=0$

Thm 3. Let $G = (V, E)$ be a digraph. Then

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$$

pf :

每增加一條 edge $u \longrightarrow v$

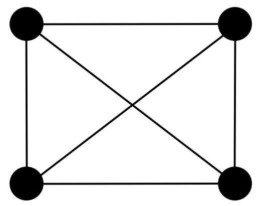
$\deg^+(u)$增加 1

$\deg^-(v)$增加 1

(ex47上面) A simple graph $G=(V, E)$ is called regular if every vertex of this graph has the same degree. A regular graph is called n-regular if $\deg(v)=n$, $\forall v \in V$.

eg.

K_4 :



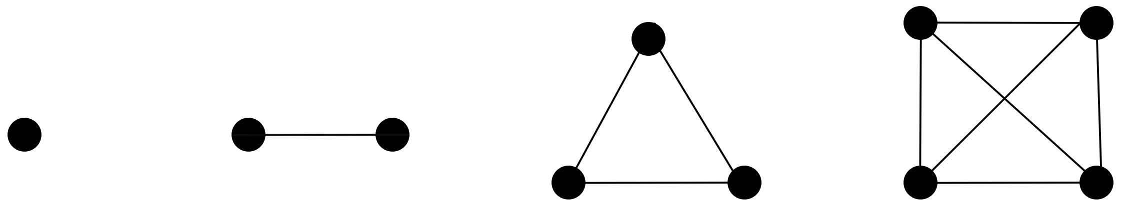
is 3-regular.

Exercise : 7, 49

Some Special Simple Graphs

Example 5.

The complete graph on n vertices, denoted by $\boldsymbol{K_n}$, is the simple graph that contains exactly one edge between each pair of distinct vertices.

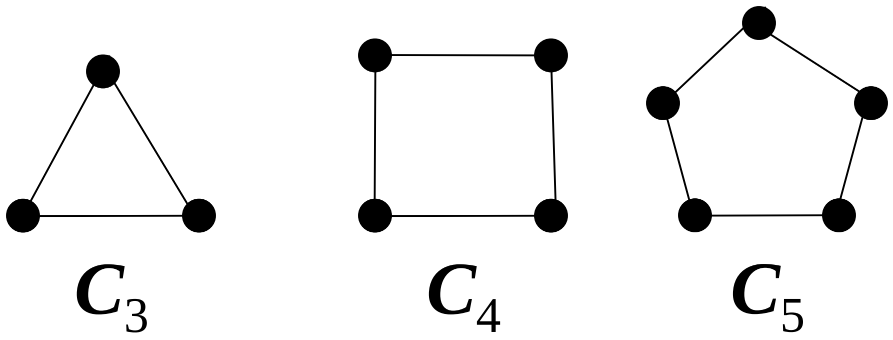


$\boldsymbol{K_1}$ $\boldsymbol{K_2}$ $\boldsymbol{K_3}$ $\boldsymbol{K_4}$

Note. K_n is $(n-1)$-regular, $|V(K_n)|=n$,

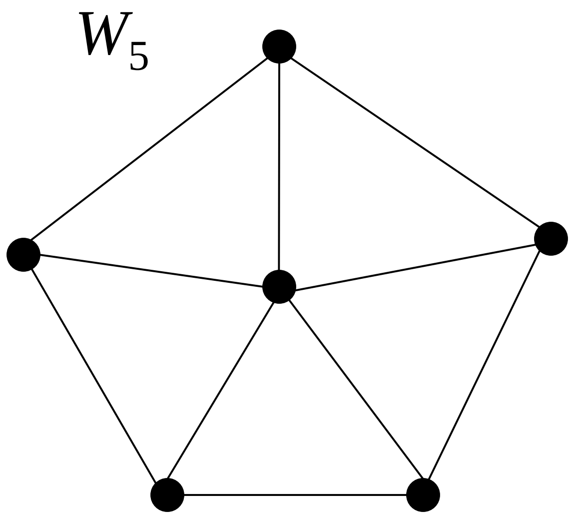
$$|E(K_n)| = \binom{n}{2}$$

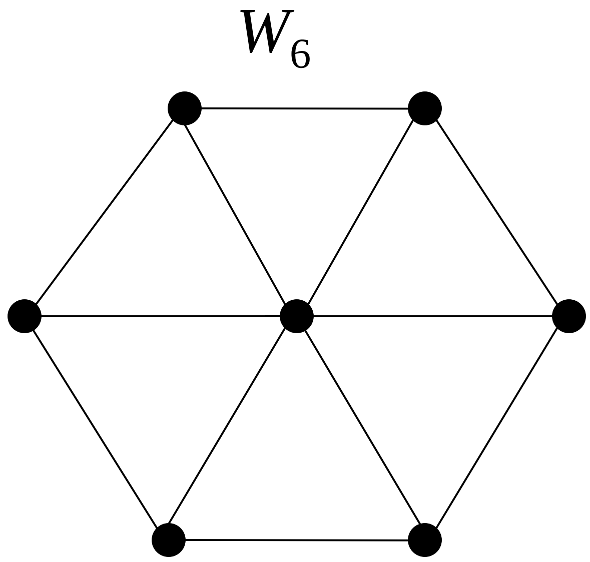
Example 6. The cycle C_n, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.



Note C_n is 2-regular, $|V(C_n)| = n$, $|E(C_n)| = n$

Example 7. We obtained the wheel W_n when we add an additional vertex to the cycle C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n, by new edges.

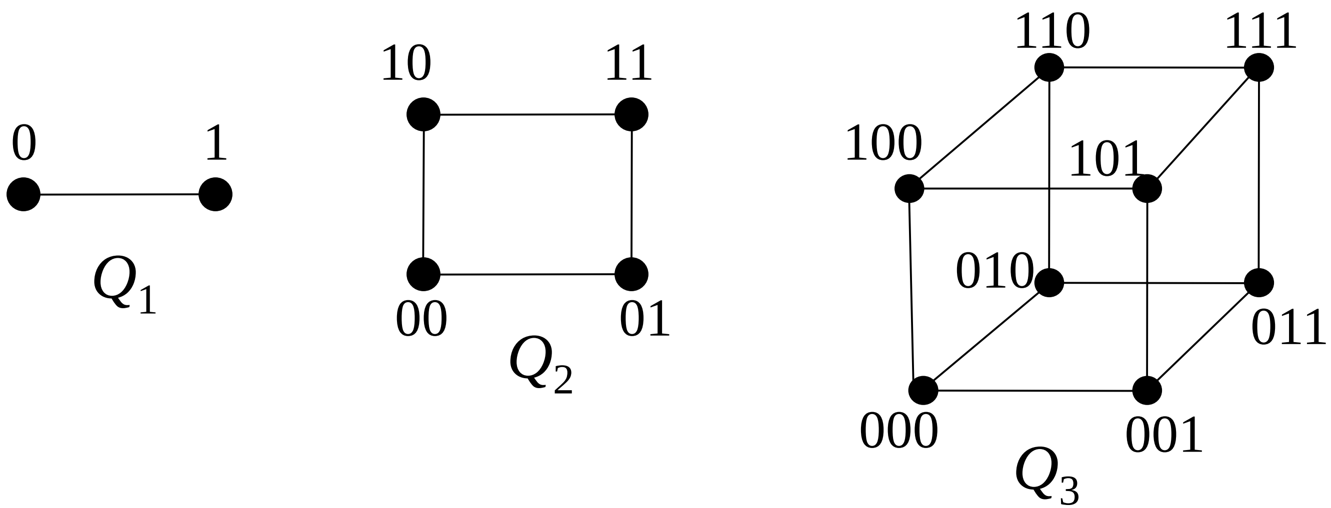




Note. $|V(W_n)| = n + 1$, $|E(W_n)| = 2n$,

W_n is not a regular graph if $n \neq 3$.

Example 8. The n-dimensional hypercube, or n-cube, denoted by Q_n, is the graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.



另法：Q_{n-1}點數
即每個dim邊數,
共n-dim

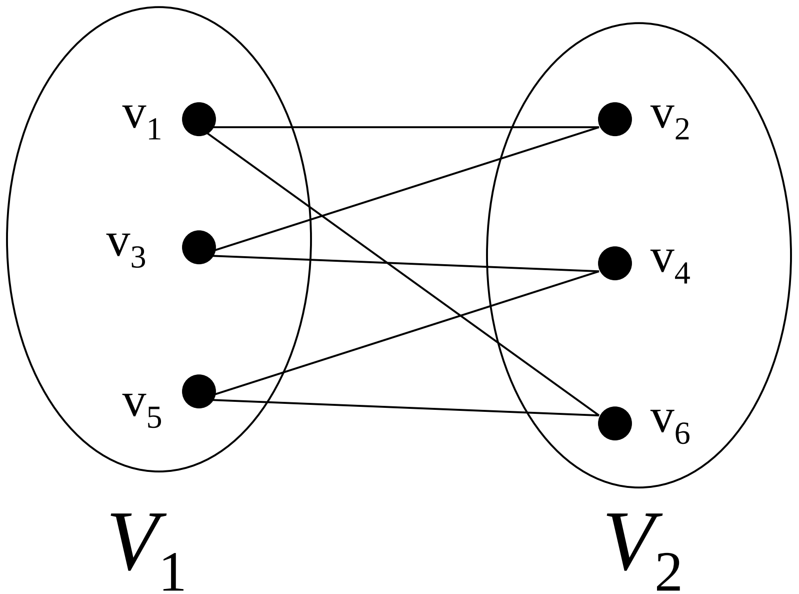
Note. Q_n is n-regular, $|V(Q_n)| = 2^n$, $|E(Q_n)| = (2^n n)/2 = 2^{n-1}n$

deg總和 / 2

Some Special Simple Graphs

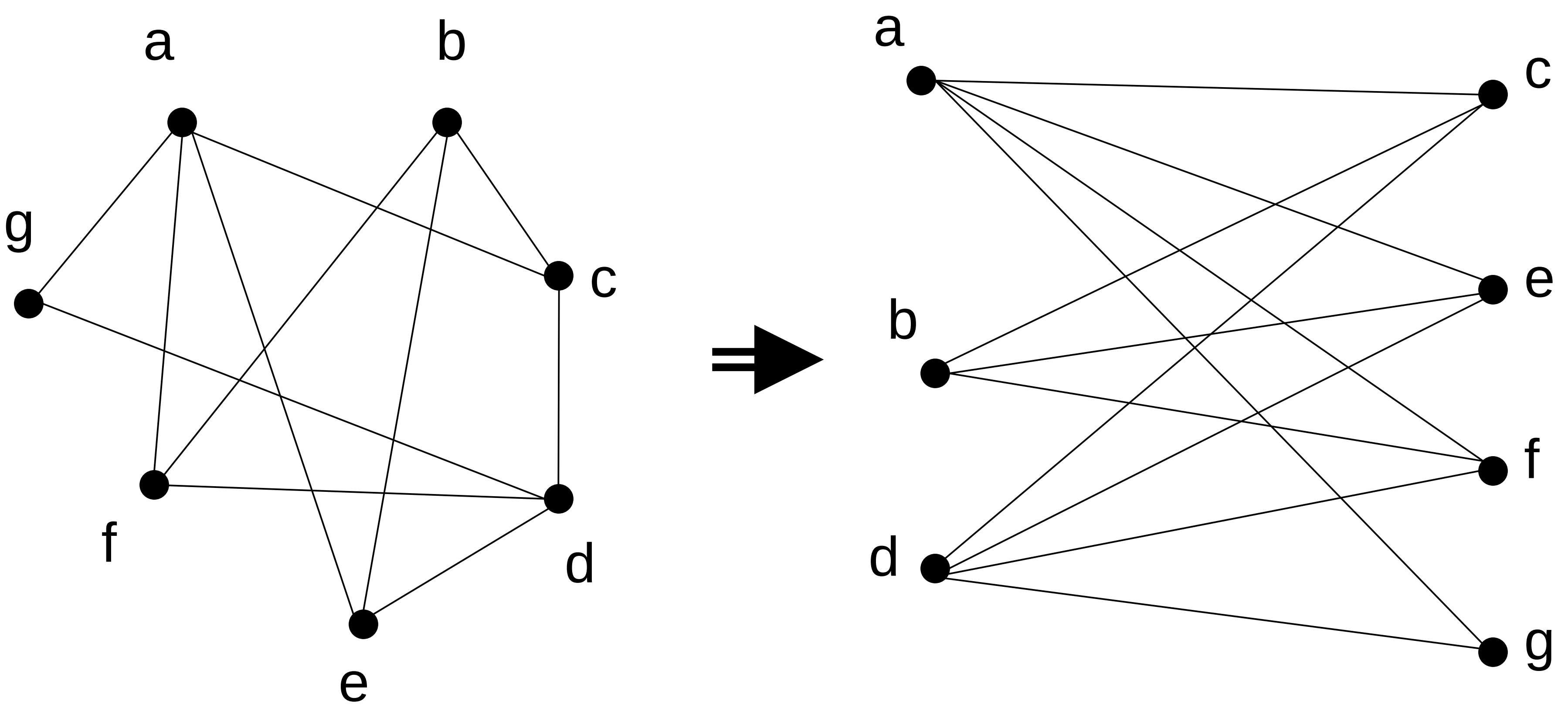
Def 5. A simple graph $G=(V,E)$ is called bipartite if V can be partitioned into V_1 and V_2, $V_1 \cap V_2 = \emptyset$, such that every edge in the graph connect a vertex in V_1 and a vertex in V_2.

Example 9.



$\therefore C_6$ is bipartite.

Example 10. Is the graph G bipartite ?

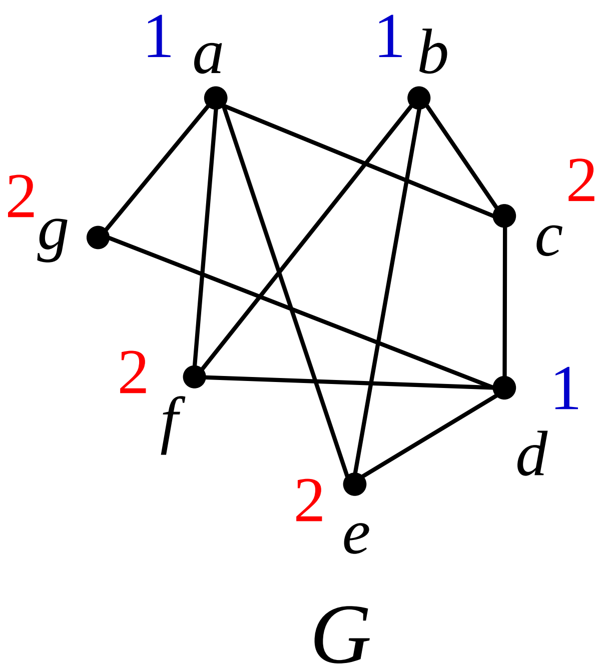


G

Yes !

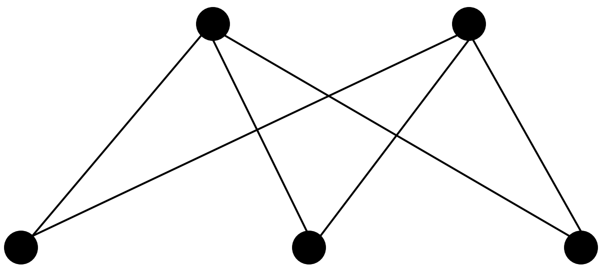
Thm 4. A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Example 12. Use Thm 4 to show that G is bipartite.

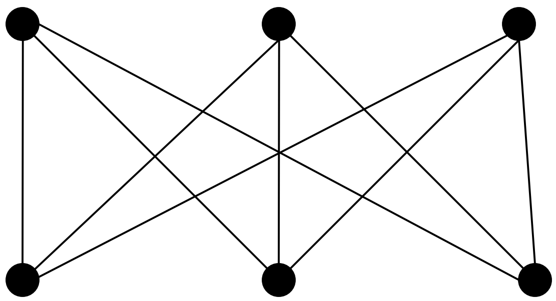


Exercise : 23, 24, 25

- Example 11. Complete Bipartite graphs ($K_{m,n}$)



$K_{2,3}$



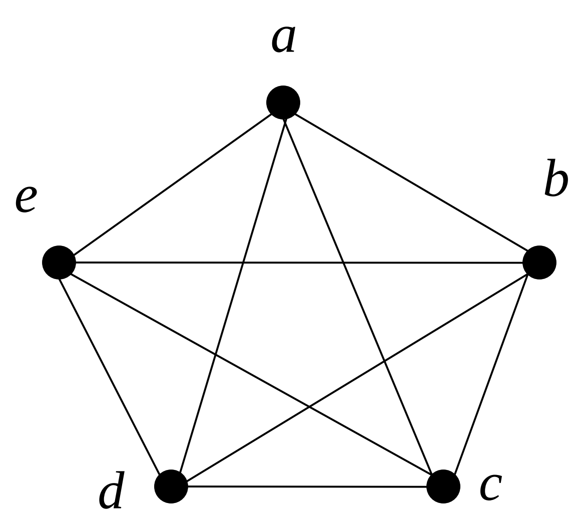
$K_{3,3}$

Note. $|V(K_{m,n})| = m+n$, $|E(K_{m,n})| = mn$,
$K_{m,n}$ is regular if and only if $m=n$.

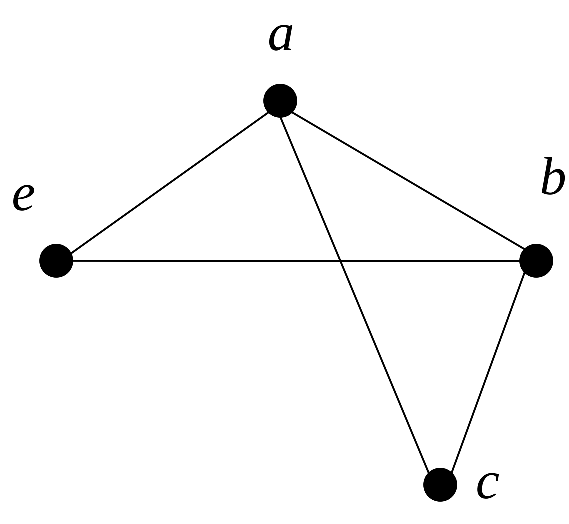
New Graphs from Old

Def 6. A subgraph of a graph $G=(V, E)$ is a graph $H=(W, F)$ where $W \subseteq V$ and $F \subseteq E$. (注意 F 要連接 W 裡的點)

Example 14. A subgraph of K_5



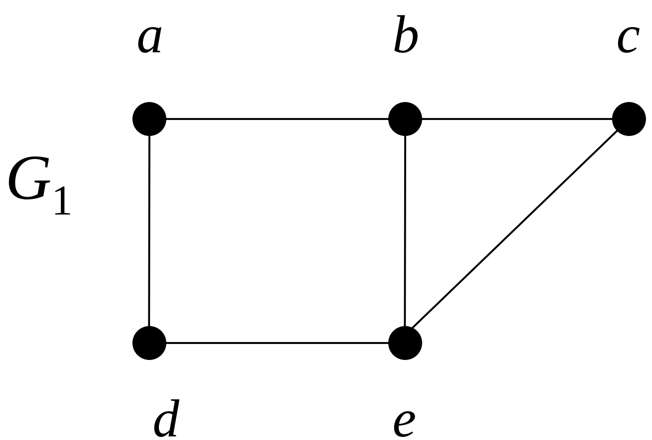
K_5

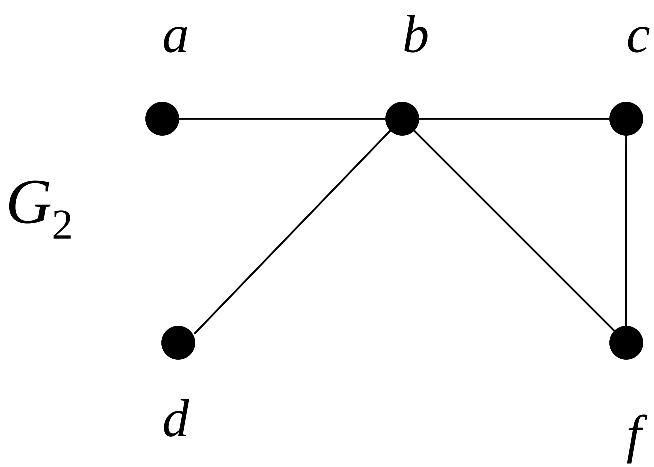


subgraph of K_5

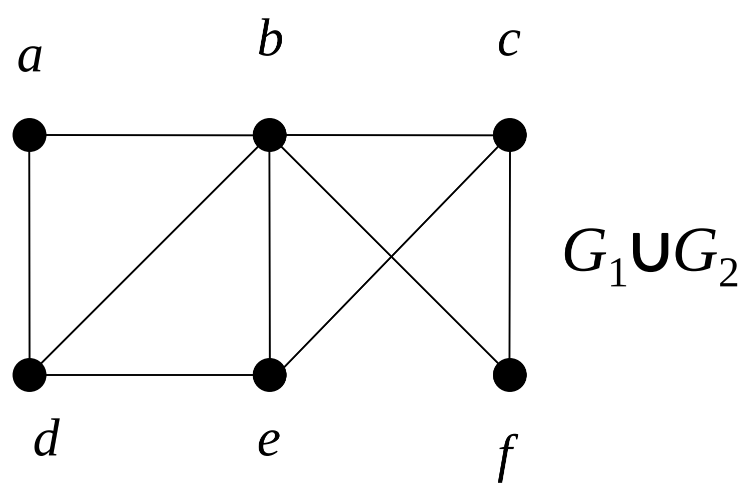
Def 7. The union of two simple graphs $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ is the simple graph $G_1 \cup G_2=(V_1 \cup V_2, E_1 \cup E_2)$

Example 15.





$\Rightarrow$

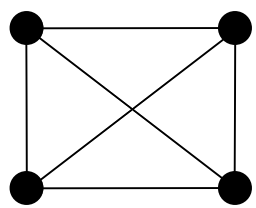


Exercise : 51

(ex35上面) A simple graph $G=(V, E)$ is called **regular** if every vertex of this graph has the same degree. A regular graph is called ***n*-regular** if $\deg(v)=n$, $\forall v \in V$.

eg.

K_4 :



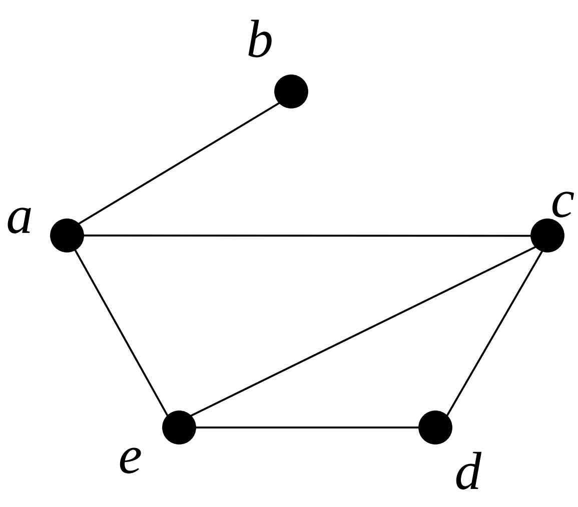
is 3-regular.

Exercise : 5, 7, 21, 23, 25, 35, 37

§9.3 Representing Graphs and Graph Isomorphism

※Adjacency list

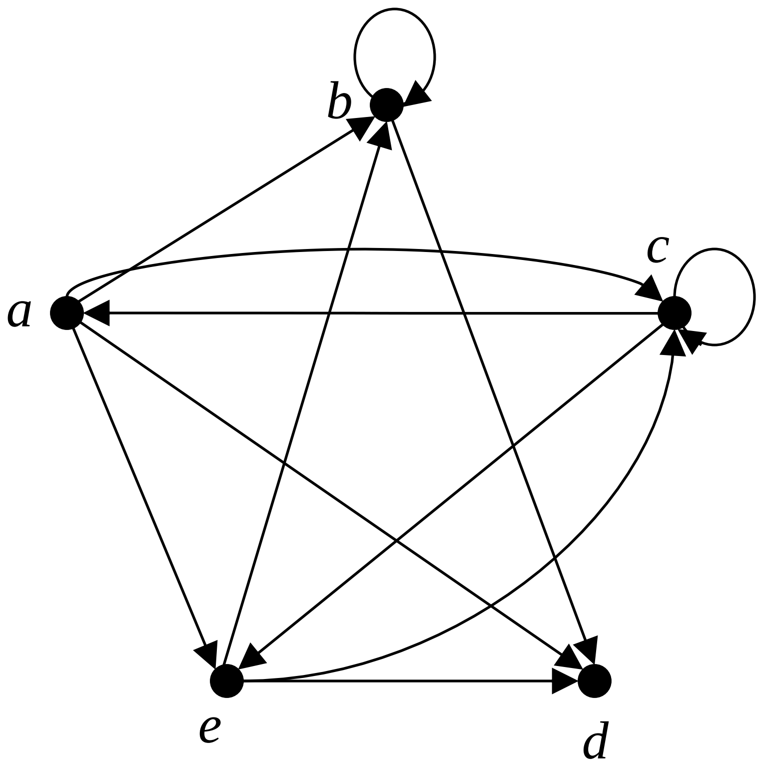
Example 1. Use adjacency lists to describe the simple graph given below.



Sol :

Vertex	Adjacent Vertices
a	*b,c,e*
b	*a*
c	*a,d,e*
d	*c,e*
e	*a,c,d*

Example 2. (digraph)



Initial vertex	Terminal vertices
a	b,c,d,e
b	b,d
c	a,c,e
d	
e	b,c,d

Exercise : 3

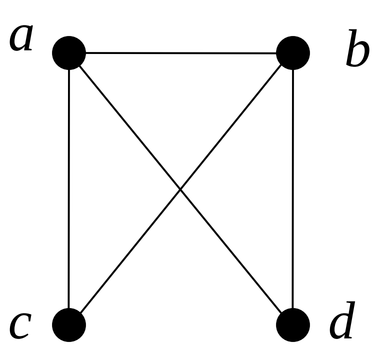
※Adjacency Matrices

Def. $G=(V, E)$: simple graph, $V=\{v_1,v_2,\ldots,v_n\}$. (順序沒關係)

A matrix A is called the adjacency matrix of G

if $A=[a_{ij}]_{n\times n}$, where $a_{ij} = \begin{cases} 1, \text{ if } \{v_i,v_j\}\in E, \\ 0, \text{ otherwise.} \end{cases}$

Example 3.

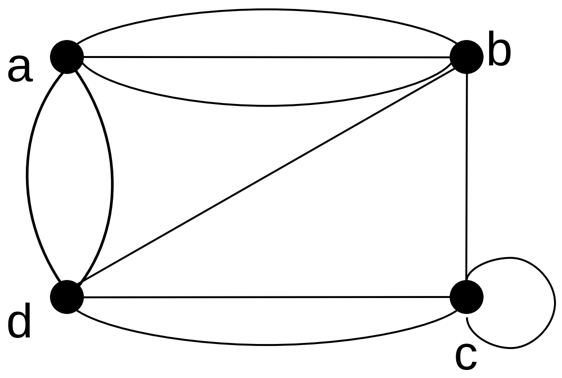


$$A_1 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array} \qquad A_2 = \begin{array}{c} \\ b \\ d \\ c \\ a \end{array} \begin{array}{c} \begin{array}{cccc} b & d & c & a \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Note:

1. There are n! different adjacency matrices for a graph with n vertices.
2. The adjacency matrix of an undirected graph is symmetric.

Example 5. (Pseudograph) (矩陣未必是0,1矩陣.)



$$\Longrightarrow \quad A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

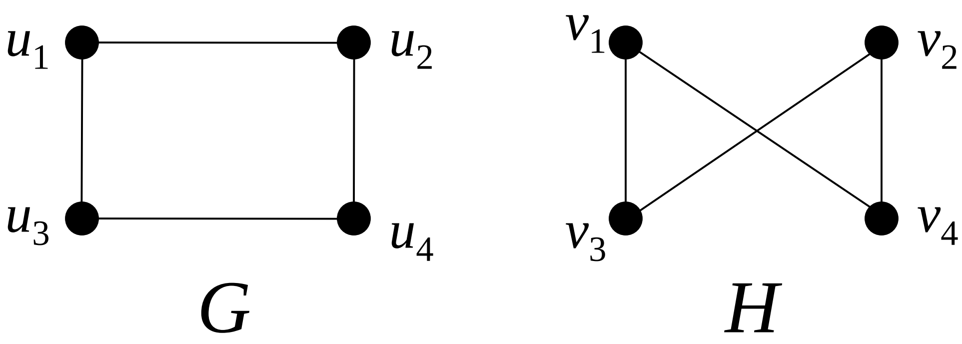
Def. If $A=[a_{ij}]$ is the adjacency matrix for the directed graph, then

$$a_{ij} = \begin{cases} 1 & \text{, if } v_i \longrightarrow v_j \\ 0 & \text{, otherwise} \end{cases}$$

(故矩陣未必對稱)

Exercise : 7, 14, 17, 23

※Isomorphism of Graphs



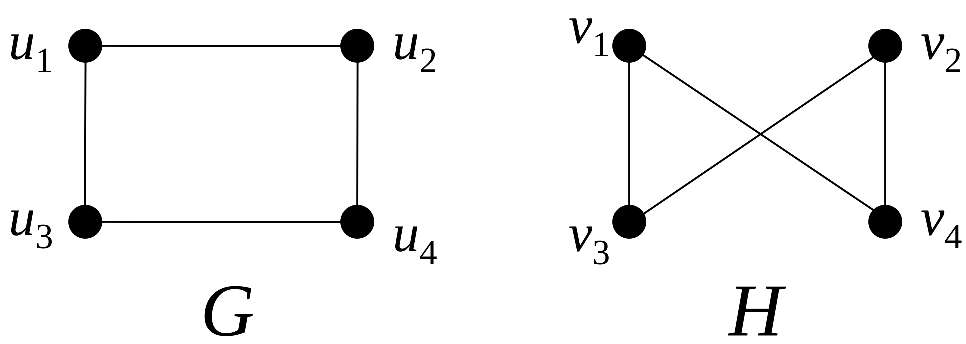
G is isomorphic to H

Def 1.

The simple graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are isomorphic if there is an one-to-one and onto function f from V_1 to V_2 with the property that $a \sim b$ in G_1 iff $f(a) \sim f(b)$ in G_2, $\forall a,b \in V_1$

f is called an isomorphism.

Example 8. Show that G and H are isomorphic.



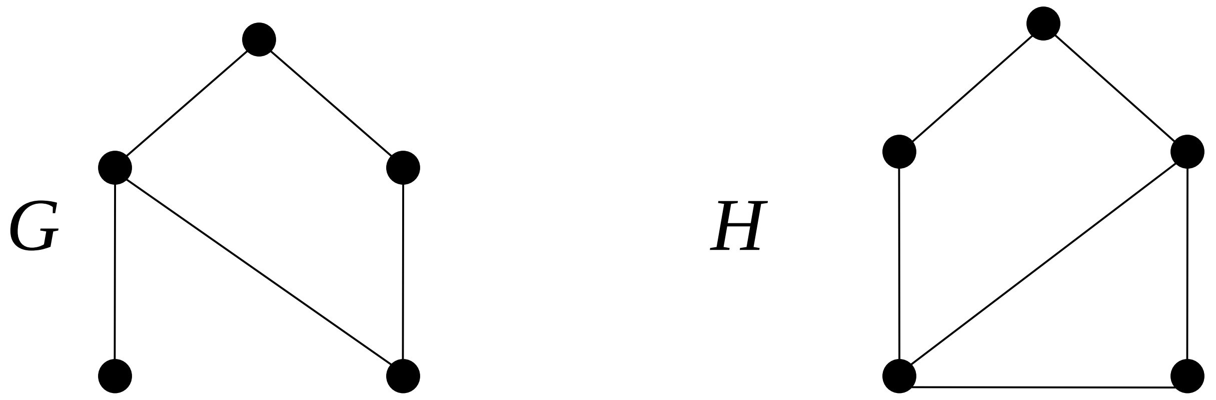
Sol.

The function f with $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, and $f(u_4) = v_2$ is a one-to-one correspondence between $V(G)$ and $V(H)$.

※Isomorphism graphs 必有 : (1) 相同的點數 (2) 相同的邊數 (3) 相同的degree分佈。

※給定二圖，判斷它們是否isomorphic的問題一般來說不易解，而且答案常是否定的。

Example 9. Show that G and H are not isomorphic.

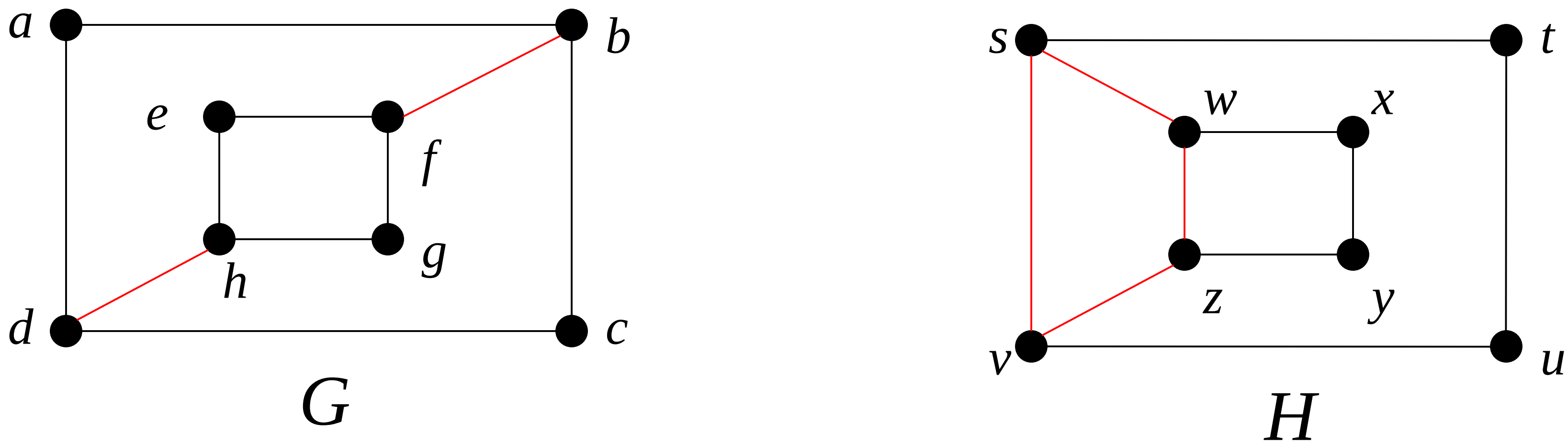


Sol :

G 有 degree $= 1$ 的點，H 沒有

Example 10.

Determine whether G and H are isomorphic.



Sol : $\because$ G 中 degree 為 3 的點有d, h, f, b

它們不能接成 4-cycle

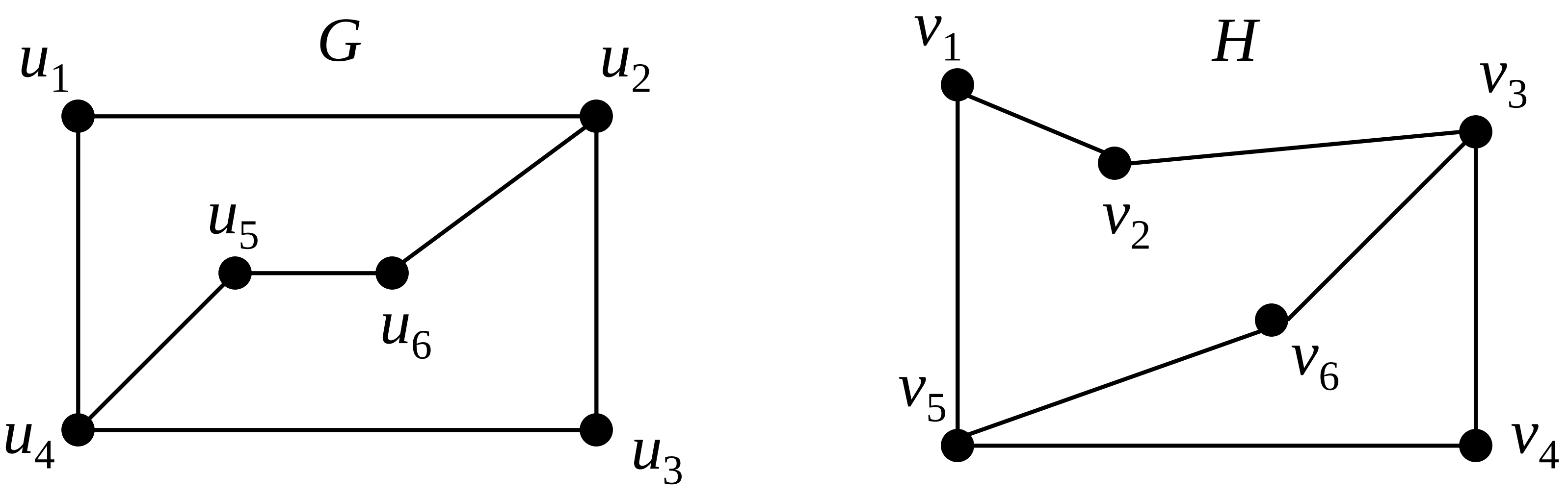
但 H 中 degree 為3的點s, w, z, v 可接成 4-cycle

$\therefore$ 不是 isomorphic.

另法 : G 中 deg = 3的點，旁邊都只連了另一個 deg = 3 的點

但 H 中 deg = 3 的點旁邊都連了 2 個 deg = 3 的點

Example 11. Determine whether the graphs G and H are isomorphic.



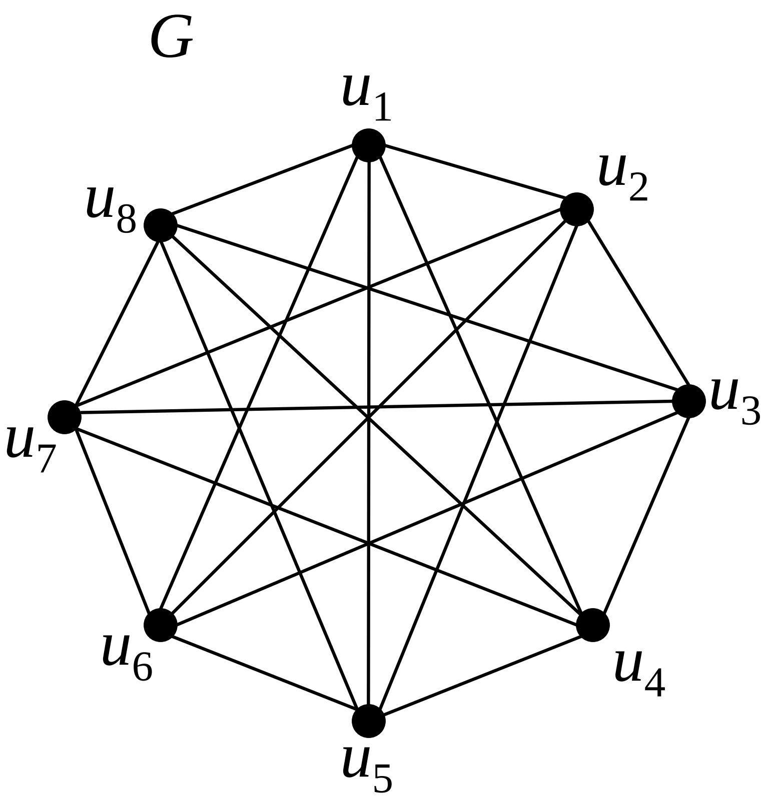
Sol:

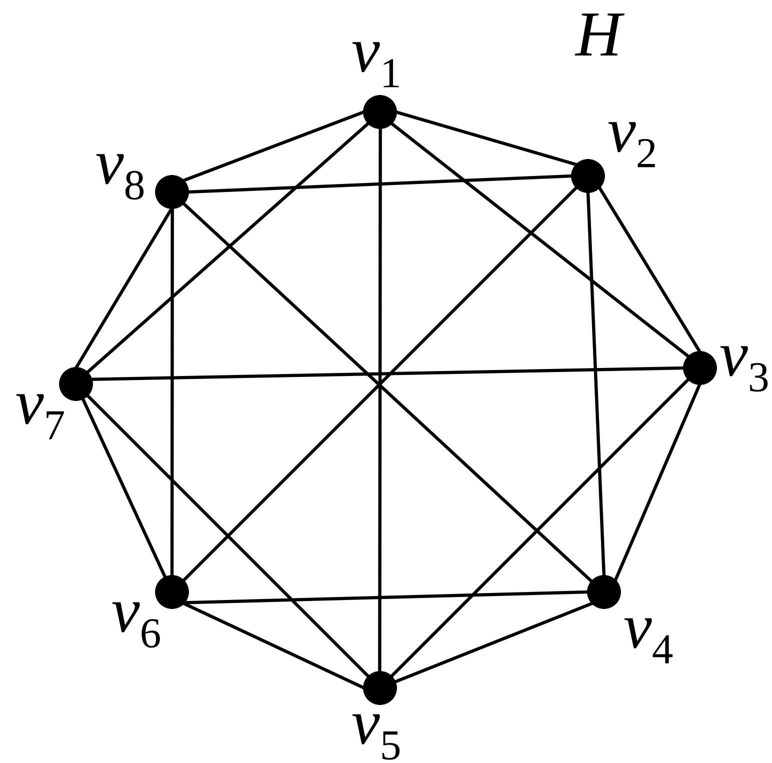
$f(u_1)=v_6$, $f(u_2)=v_3$, $f(u_3)=v_4$, $f(u_4)=v_5$, $f(u_5)=v_1$, $f(u_6)=v_2$

$\Rightarrow$ Yes

Exercise : 37, 39, 40

Ex44. Determine whether the graphs G and H are isomorphic.





此二圖都是vertex-transitive，每個點的角色是一樣的，故考慮u_1及v_1即可

v_1的5 個neighbor可以連接成5-cycle，但u_1的不行，故不是isomorphic

§9.4: Connectivity

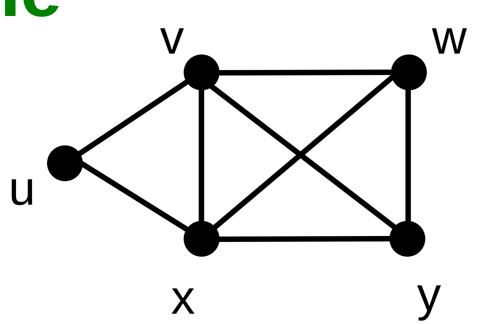
Def. 1,2 :

In an undirected graph, a path of length n from u to v is a sequence of $n+1$ adjacent vertices going from vertex u to vertex v.
(e.g., P: $u=x_0, x_1, x_2, \ldots, x_n=v$.) (P has n edges.)

Def: (與書上的不太一致)

path中的點及邊不可重複
trail: 允許點重複的路徑 (邊不可重複)
walk: 允許點及邊重複的路徑

Example



path: u, v, y
trail: u, v, w, y, v, x, y
walk: u, v, w, v, x, v, y

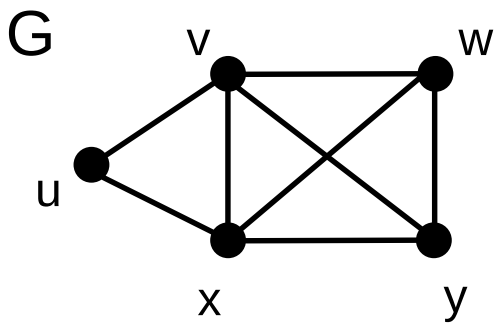
Def:

cycle: path with $u=v$

circuit: trail with $u=v$

closed walk: walk with $u=v$

Example



cycle: u, v, y, x, u

trail: u, v, w, y, v, x, u

walk: u, v, w, v, x, v, y, x, u

Exercise : 17

Paths in Directed Graphs

The same as in undirected graphs, but the path must go in the direction of the arrows.

Connectedness in Undirected Graphs

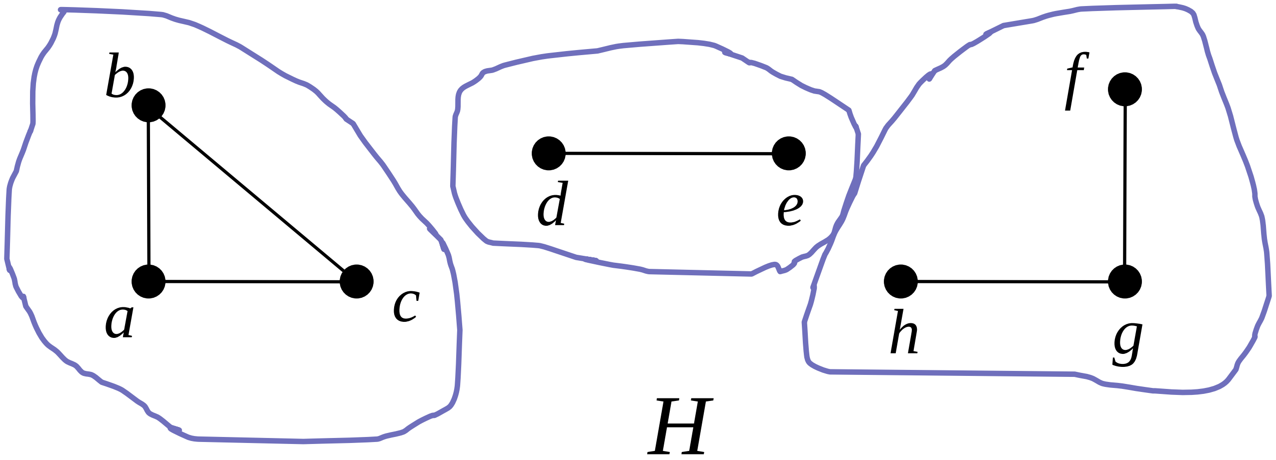
Def. 3:

An undirected graph is *connected* (連通) if there is a path between every pair of distinct vertices in the graph.

Def:

Connected component: maximal connected subgraph. (一個不連通的圖會有好幾個component)

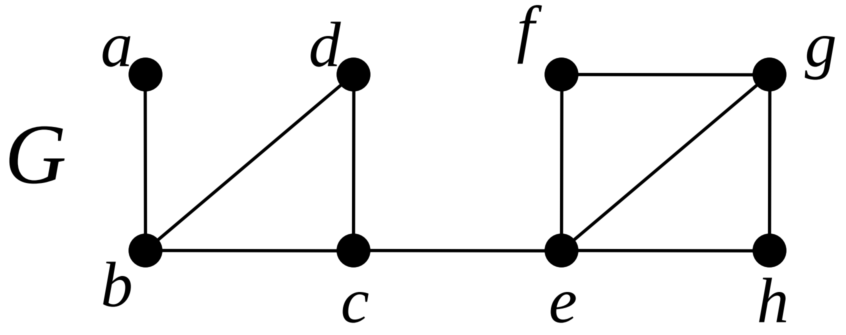
Example 6 What are the connected components of the graph H?



Def:

A *cut vertex* separates one connected component into several components if it is removed.
A *cut edge* separates one connected component into two components if it is removed.

Example 8. Find the cut vertices and cut edges in the graph G.



Sol:

cut vertices: b, c, e

cut edges:

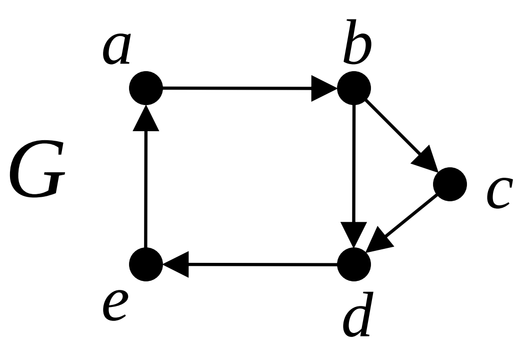
$\{a, b\}, \{c, e\}$

Exercise : 29,31, 32

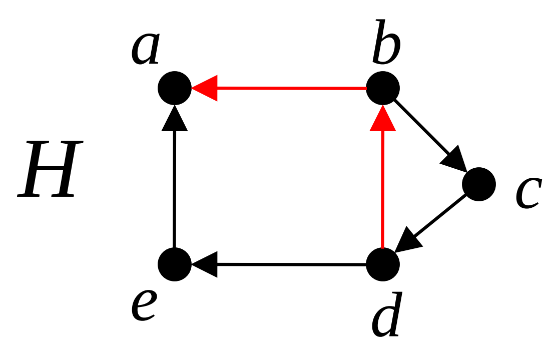
Connectedness in Directed Graphs

Def. 4: A directed graph is *strongly connected* if there is a path from a to b for any two vertices a, b. A directed graph is *weakly connected* if there is a path between every two vertices in the underlying undirected graphs.

Example 9 Are the directed graphs G and H strongly connected or weakly connected?



strongly connected

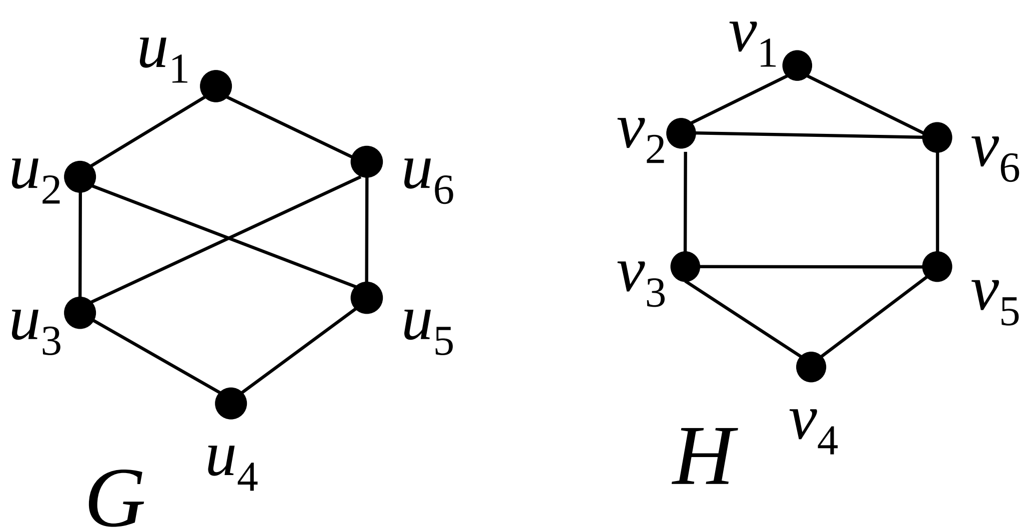


weakly connected

Paths and Isomorphism

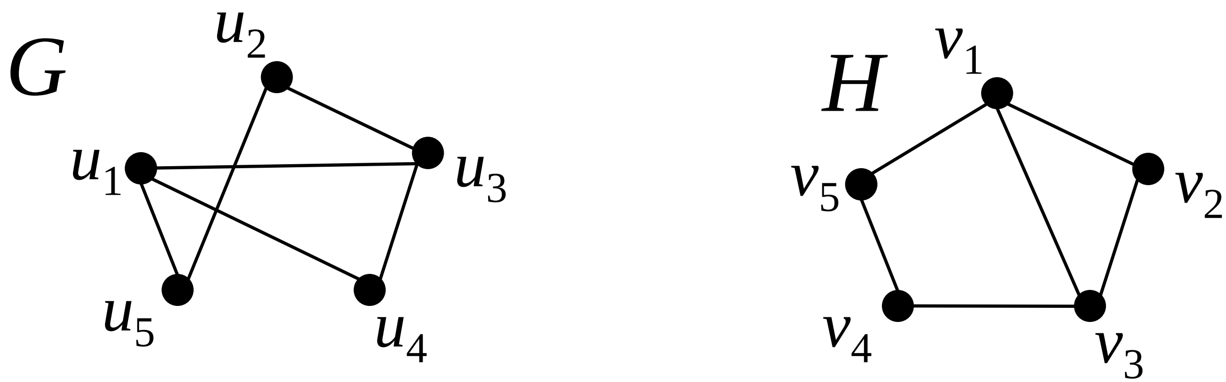
Note that connectedness, and the existence of a circuit or simple circuit of length k are graph invariants with respect to isomorphism.

Example 12. Determine whether the graphs G and H are isomorphic.



Sol: No, 因G沒有三角形，H有

Example 13. Determine whether the graphs G and H are isomorphic.



Sol.

Both G and H have 5 vertices, 6 edges, two vertices of deg 3, three vertices of deg 2, a 3-cycle, a 4-cycle, and a 5-cycle. $\Rightarrow$ G and H may be isomorphic.

The function f with $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, $f(u_4) = v_2$ and $f(u_5) = v_5$ is a one-to-one correspondence between $V(G)$ and $V(H)$. $\Rightarrow$ G and H are isomorphic.

Exercise : 19, 20

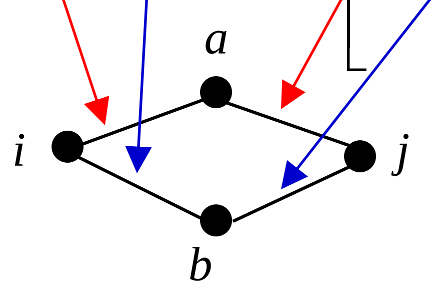
Counting Paths between Vertices

Theorem 2:

Let G be a graph with adjacency matrix A with respect to the ordering $v_1, v_2, \ldots, v_n$. The number of walks of length r from v_i to v_j is equal to $(A^r)_{i,j}$.

Proof (僅簡單舉例說明)

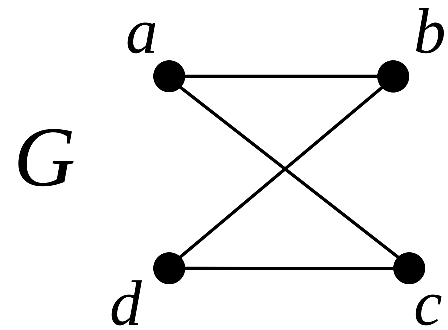
$$\overset{A}{\begin{array}{cc} & \begin{array}{ccccc} & a & & b & \end{array} \\ i & \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & 1 & \cdots & 1 & \cdots \\ & \vdots & & \vdots & \end{bmatrix} \end{array}}
\overset{A}{\begin{array}{cc} & \begin{array}{ccc} & j & \end{array} \\ \begin{array}{c} \\ a \\ \\ b \\ \\ \end{array} & \begin{bmatrix} & \vdots & \\ \cdots & 1 & \cdots \\ & \vdots & \\ \cdots & 1 & \cdots \\ & \vdots & \end{bmatrix} \end{array}}
=
\overset{A^2}{\begin{array}{cc} & \begin{array}{ccc} & j & \end{array} \\ i & \begin{bmatrix} & \vdots & \\ \cdots & 2 & \cdots \\ & \vdots & \end{bmatrix} \end{array}}$$



Example 14. How many walks of length 4 are there from a to d in the graph G?

Sol.

The adjacency matrix of G (ordering as a, b, c, d) is



$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \Rightarrow A^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix} \Rightarrow 8$$

Q: 哪8條？

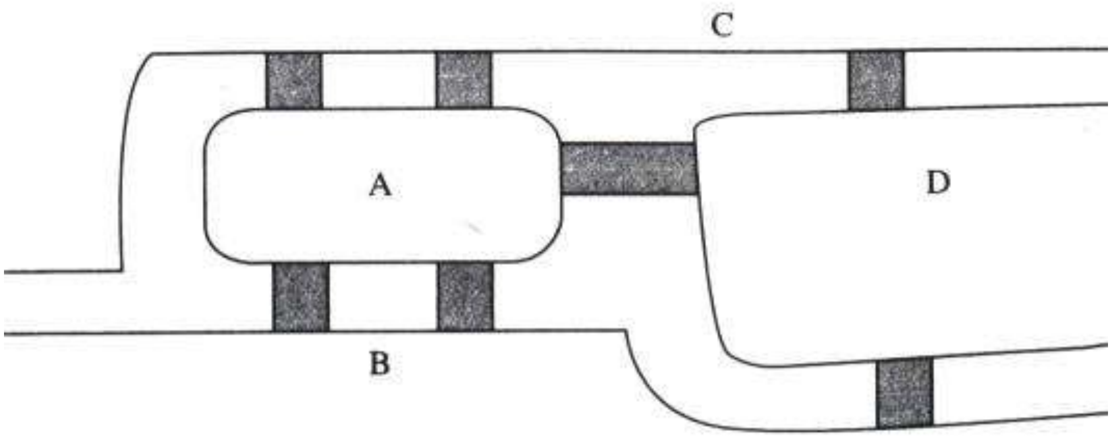
a-b-a-b-d, a-b-a-c-d, a-c-a-b-d, a-c-a-c-d,
a-b-d-b-d, a-b-d-c-d, a-c-d-b-d, a-c-d-c-d

Exercise : 17

§9.5: Euler & Hamilton Paths

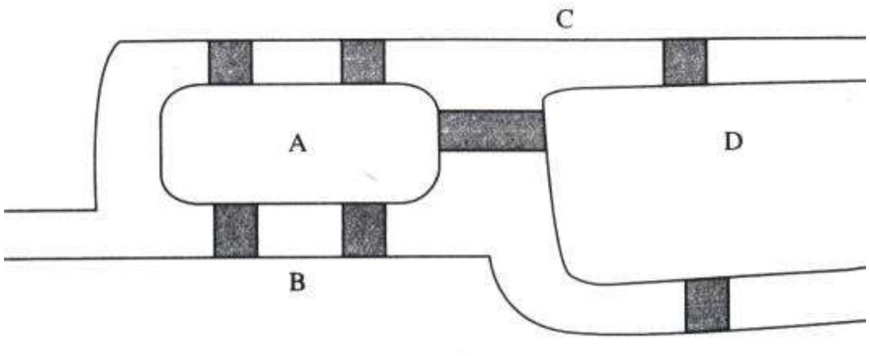
Graph Theory 的起源

- 1736, Euler solved the Königsberg Bridge Problem (七橋問題)

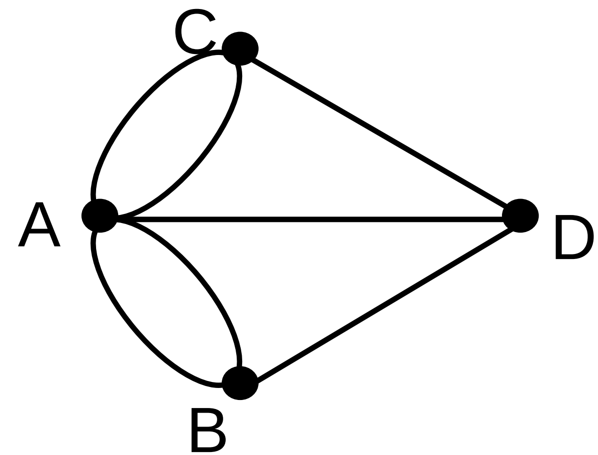


Q: 是否存在一種走法，可以走過每座橋一次，並回到起點？

Königsberg Bridge Problem



⇒



Q: 是否存在一種走法，可以走過每條邊一次，並回到起點？

Ans: 因為每次經過一個點，都需要從一條邊進入該點，再用另一條邊離開，所以經過每個點一次要使用掉一對邊。

⇒ 每個點上連接的邊數必須是偶數才行

⇒ 此種走法不存在

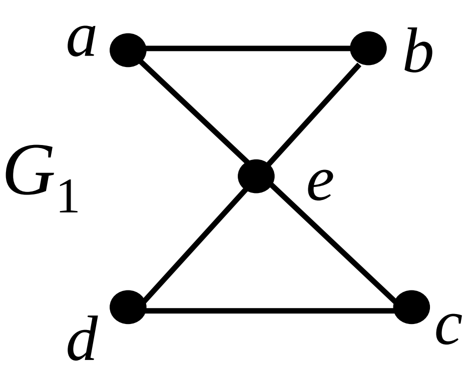
Def 1:

An *Euler circuit* in a graph *G* is a simple circuit containing every edge of *G*.
An *Euler path* in *G* is a simple path containing every edge of *G*.

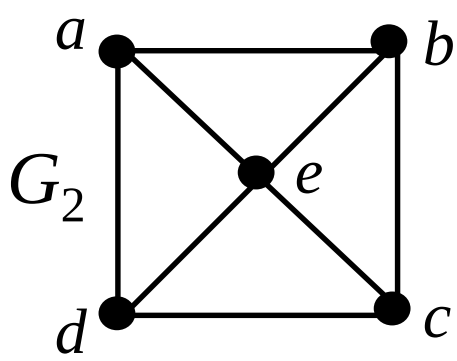
Thm. 1:
A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

Thm. 2:
A connected multigraph has an Euler path (but not an Euler circuit) if and only if it has exactly 2 vertices of odd degree.

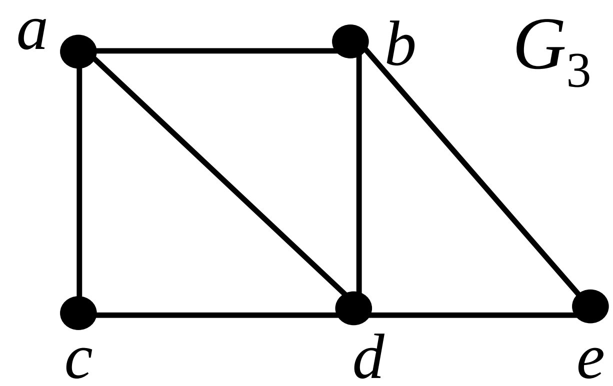
Example 1. Which of the following graphs have an Euler circuit or an Euler path?



Euler circuit



none



Euler path

Exercise : 3, 5, 21, 26, 28

Algorithm 1 (Constructing Euler Circuits.)

Procedure *Euler*(*G*: connected multigraph with all vertices of even degree)

circuit := a circuit in *G* beginning at an arbitrary chosen vertex with edges successively added to form a path that returns to this vertex

H := *G* with the edges of this circuit removed

while *H* has edges

begin

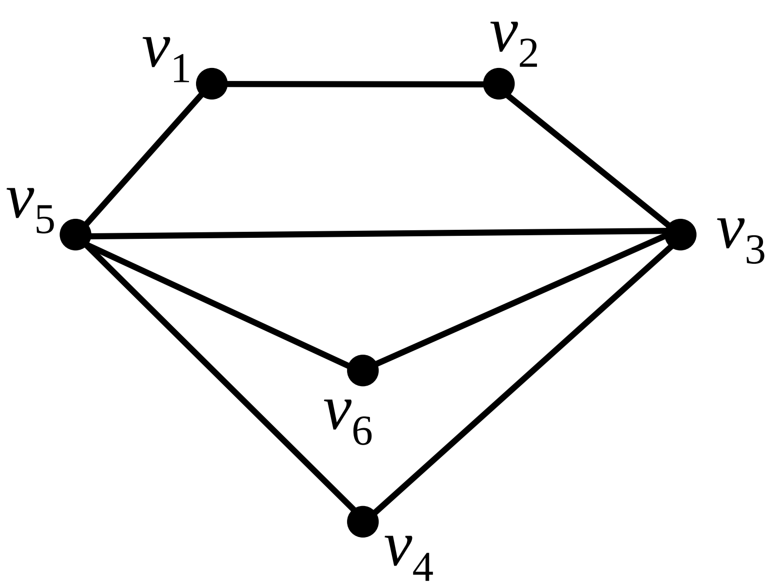
subcircuit := a circuit in *H* beginning at a vertex in *H* that also is an endpoint of an edge of *circuit*

H := *H* with edges of *subcircuit* and all isolated vertices removed

circuit := *circuit* with *subcircuit* inserted at appropriate vertex

end {*circuit* is an Euler circuit}

Example



Step 1: find the 1st circuit

C: $v_1, v_2, v_3, v_4, v_5, v_1$

Step 2: $H = G - C \neq \varnothing$,
find subcircuit

SC: v_3, v_5, v_6, v_3

Step 3:

$C = C \cup SC$,

$H = G - C = \varnothing$, stop

C: $v_1, v_2, \underbrace{v_3, v_5, v_6, v_3}_{SC\text{ embedded}}, v_4, v_5, v_1$

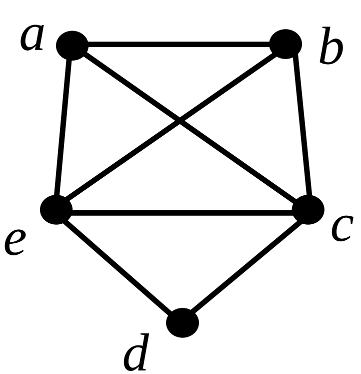
Exercise : 7

Hamilton Paths and Circuits

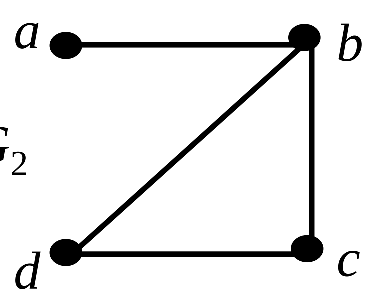
Def. 2: A *Hamilton path* is a path that traverses each vertex in a graph G exactly once.
A *Hamilton circuit* is a circuit that traverses each vertex in G exactly once.

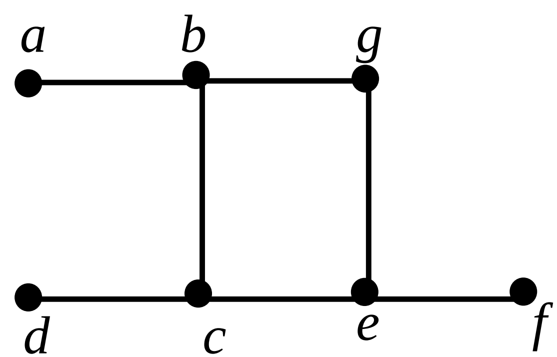
Example 1. Which of the following graphs have a Hamilton circuit or a Hamilton path?

G_1



G_2





G_3

Exercise : 42, 43

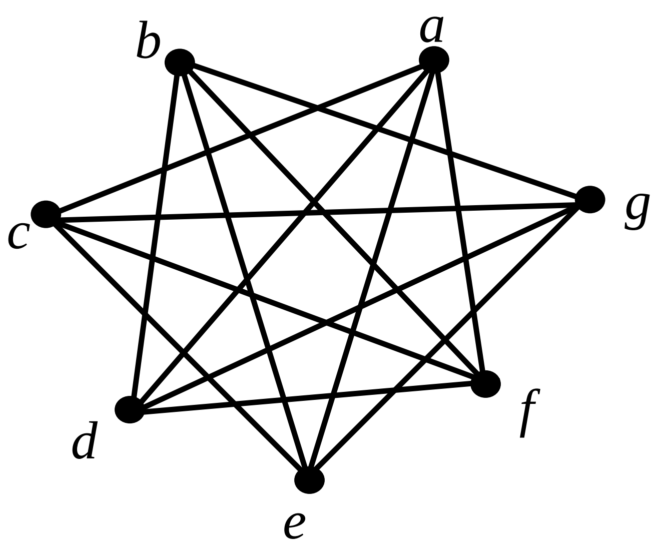
Hamilton circuit: G_1

Hamilton path: G_1, G_2

Thm. 3 (Dirac's Thm.):

If (but <u>not</u> only if) G is a simple graph with $n \geq 3$ vertices such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.

Example



each vertex has deg $\geq n/2 = 3.5$

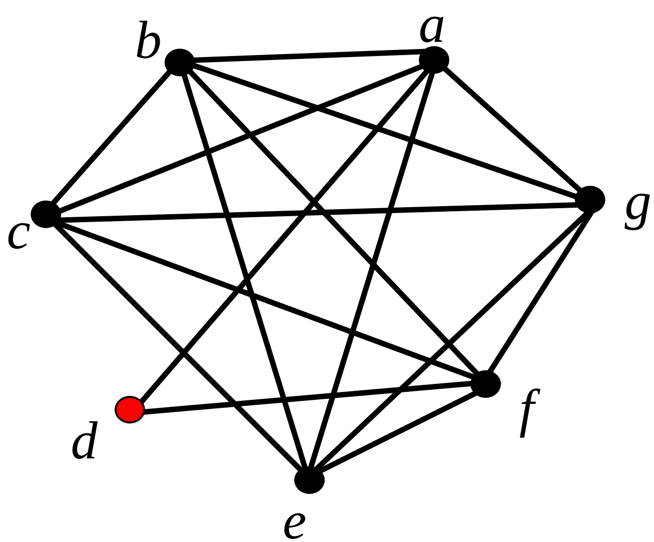
$\Rightarrow$Hamilton circuit exists

如： a, c, e, g, b, d, f, a

Thm. 4 (Ore's Thm.):

If G is a simple graph with $n \geq 3$ vertices such that $\deg(u)+\deg(v) \geq n$ for every pair of nonadjacent vertices u and v, then G has a Hamilton circuit.

Example



each nonadjacent vertex pair has deg sum $\geq n = 7$

$\Rightarrow$Hamilton circuit exists

如： a, d, f, e, c, b, g, a

§9.6: Shortest-Path Problems

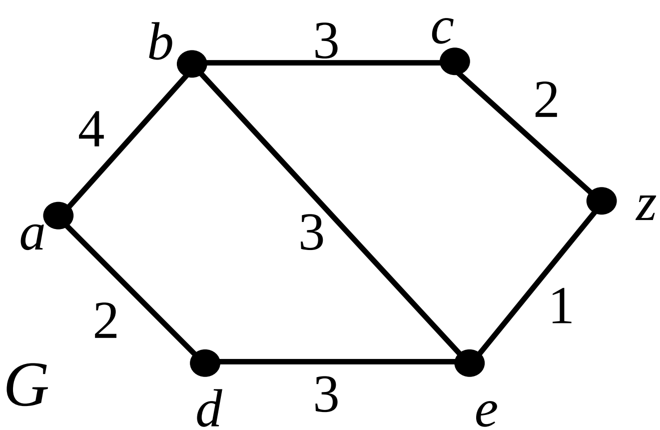
Def:

1. Graphs that have a number assigned to each edge are called *weighted graphs*.
2. The length of a path in a weighted graph is the sum of the weights of the edges of this path.

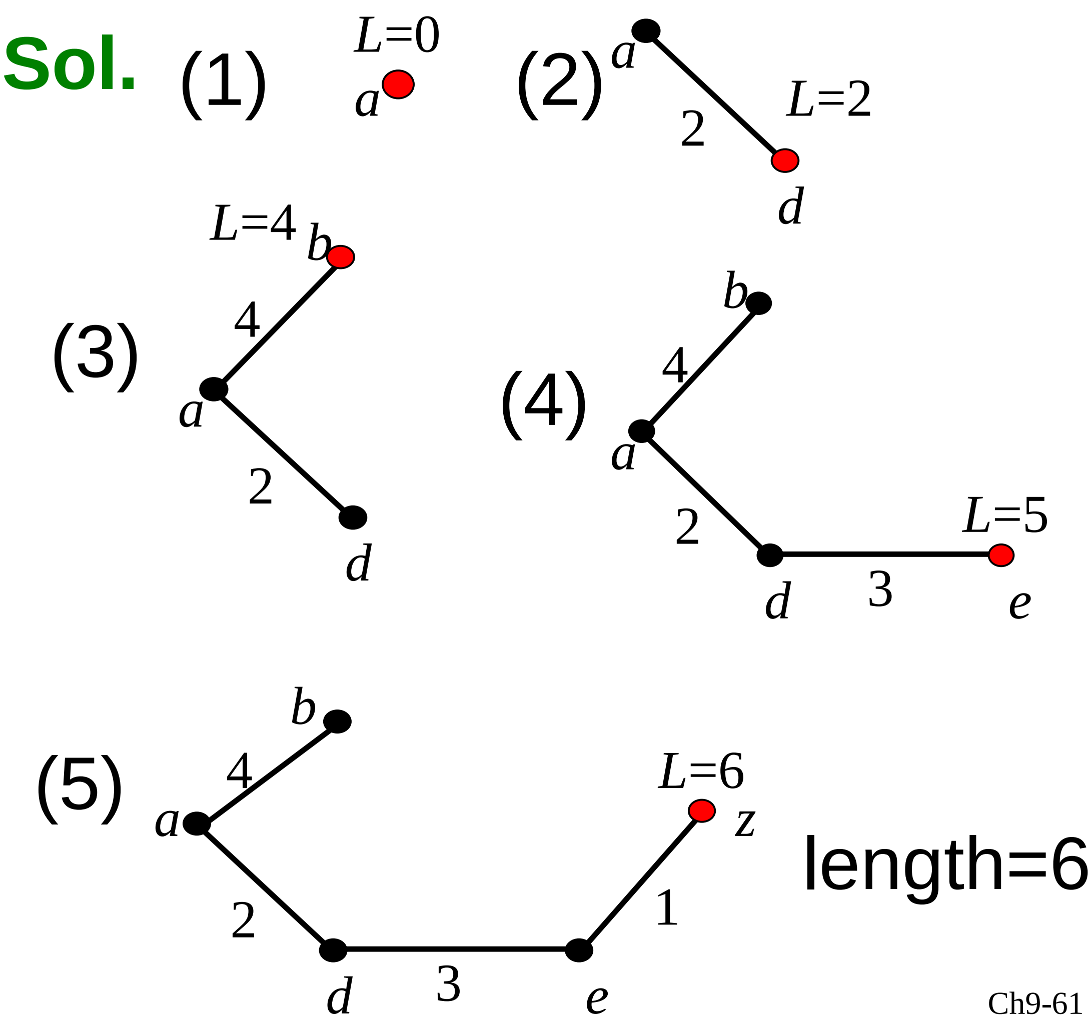
Shortest path Problem:

Determining the path of least sum of the weights between two vertices in a weighted graph.

Example 1. What is the length of a shortest path between a and z in the weighted graph G?



Sol.



每個step都從還未處理的點找離a最近的，此時a至該點的shortest path就找到了。

Dijkstra's Algorithm (find the length of a shortest path from a to z)

Procedure *Dijkstra*(G: weighted connected simple graph, with all weights positive)

{G has vertices $a = v_0, v_1, \ldots, v_n = z$ and weights $w(v_i, v_j)$ where $w(v_i, v_j) = \infty$ if $\{v_i, v_j\}$ is not an edge in G}

for $i := 1$ **to** n

  $L(v_i) := \infty$

$L(a) := 0$

$S := \varnothing$

while $z \notin S$

begin

  $u :=$ a vertex not in S with $L(u)$ minimal

  $S := S \cup \{u\}$

  for all vertices v not in S

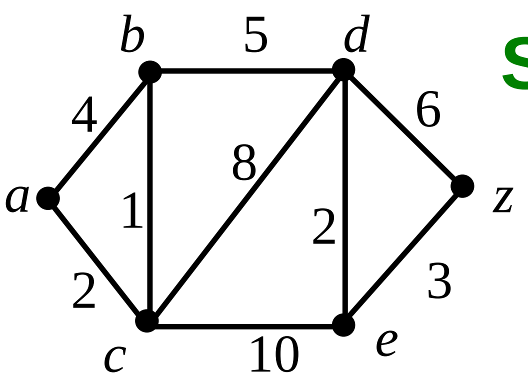
    if $L(\mathrm{u}) + w(u, v) < L(v)$ **then** $L(v) := L(\mathrm{u}) + w(u, v)$

end {$L(z)$ = length of a shortest path from a to z}

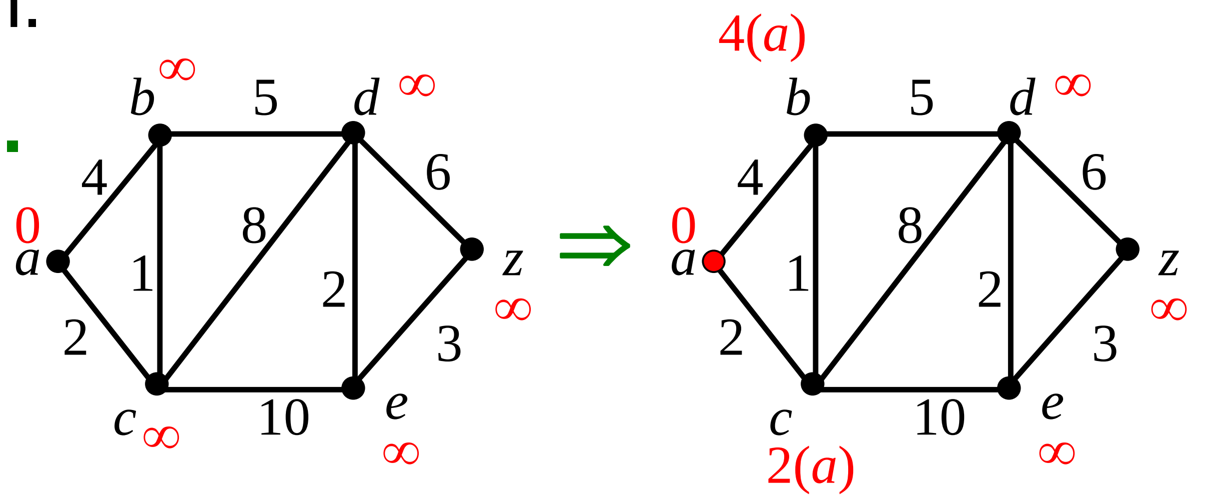
This algorithm can be extended to construct a shortest path.

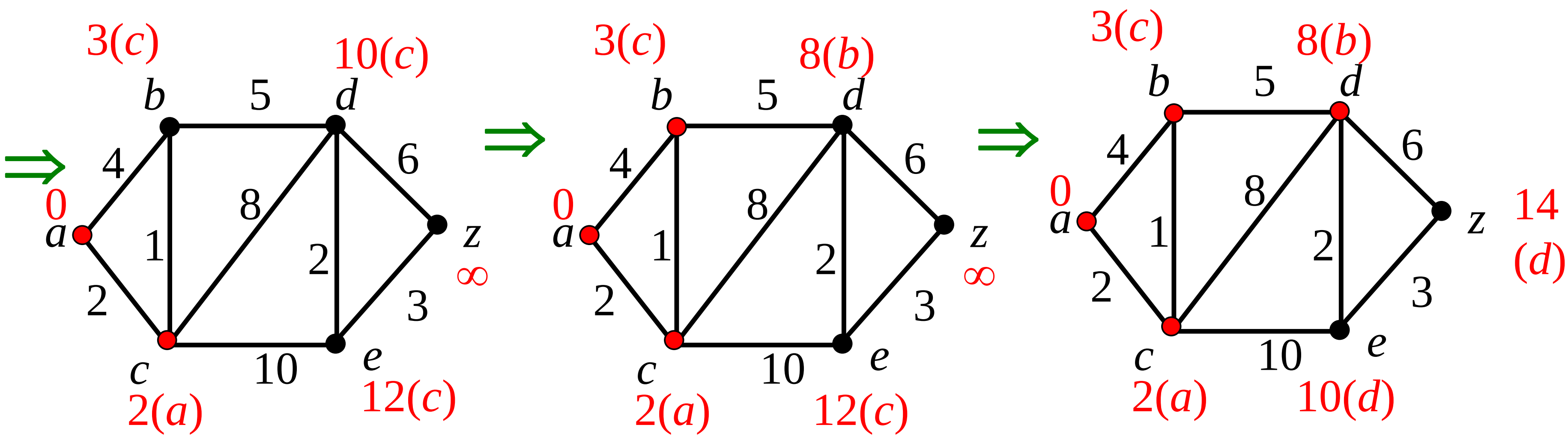
此處加一個變數
紀錄 v 的前一點是 u
previous(v) := u
演算法最後再從 z 往前trace

Example 2. Use Dijkstra's algorithm to find the length of a shortest path between a and z in the weighted graph.

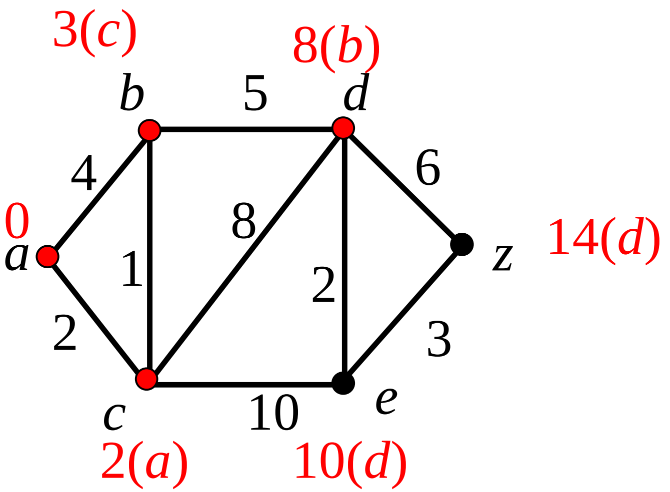


Sol.

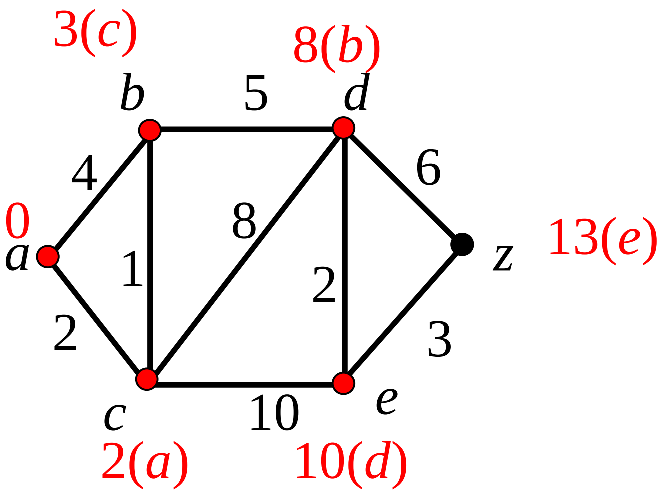




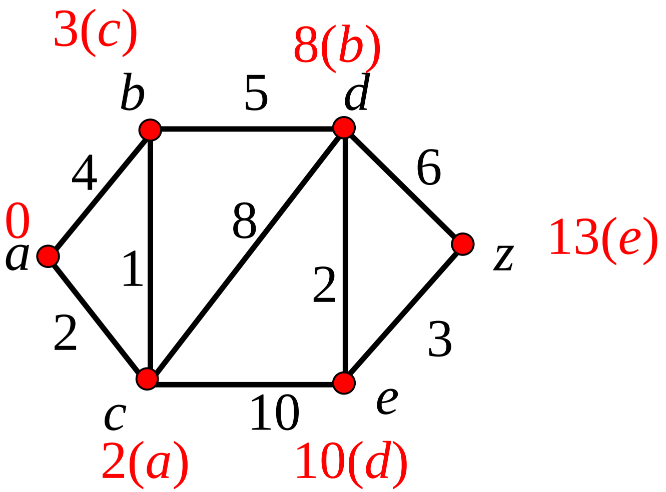
(前頁最後一個圖)



⇒



⇒



⇒ path: a, c, b, d, e, z

length: 13

Exercise : 3

Thm. 1

Dijkstra's algorithm finds the length of a shortest path between two vertices in a connected simple undirected weighted graph.

Thm. 2

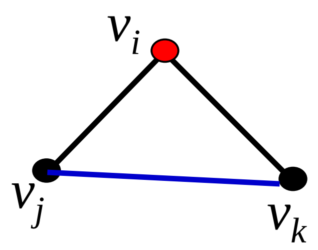
Dijkstra's algorithm uses $O(n^2)$ operations (additions and comparisons) to find the length of a shortest path between two vertices in a connected simple undirected weighted graph with n vertices.

Floyd's Algorithm (find the distance $d(a, b)$ $\forall a, b$)

Procedure *Floyd*(G: weighted simple graph)

{G has vertices $v_1, \ldots, v_n$ and weights $w(v_i, v_j)$ with
$w(v_i, v_j) = \infty$ if $\{v_i, v_j\}$ is not an edge}

for i := 1 **to** n
  for j := 1 **to** n
    $d(v_i, v_j) := w(v_i, v_j)$
for i := 1 **to** n
  for j := 1 **to** n
    for k := 1 **to** n
      if $d(v_j, v_i) + d(v_i, v_k) < d(v_j, v_k)$
        then $d(v_j, v_k) := d(v_j, v_i) + d(v_i, v_k)$
{$d(v_i, v_j)$ is the length of a shortest path between v_i and v_j}



對每個v_i， 檢查任何一對點 v_j, v_k 目前的距離會不會因為改走 v_i 而變小。

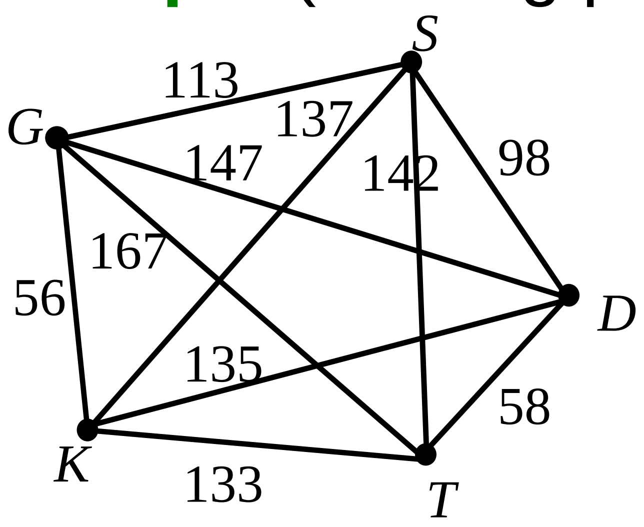
This algorithm cannot be used to construct shortest paths.

Exercise : 21

The Traveling Salesman Problem: (簡介)

A traveling salesman wants to visit each of n cities exactly once and return to his starting point. In which order should he visit these cities to travel the minimum total distance?

Example (starting point D)



$D \to T \to K \to G \to S \to D$: 458

$D \to T \to S \to G \to K \to D$: 504

$D \to T \to S \to K \to G \to D$: 540

…

相當於在complete weighted graph 上找 minimum weight Hamiltonian circuit.

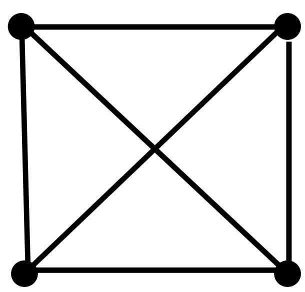
不存在polynomial time alg. 效率好的 algorithm 只能找近似解

§9.7: Planar Graphs

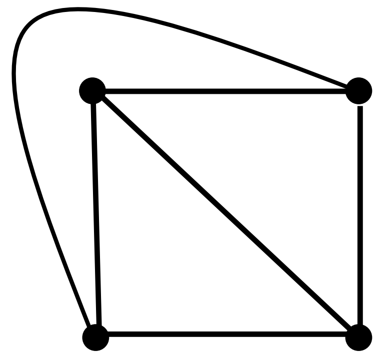
Def 1.

A graph is called *planar* if it can be drawn in the plane without any edge crossing. Such a drawing is called a *planar representation* of the graph.

Example 1: Is K_4 planar?



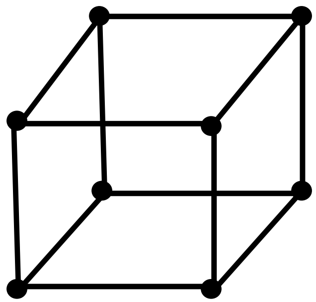
K_4



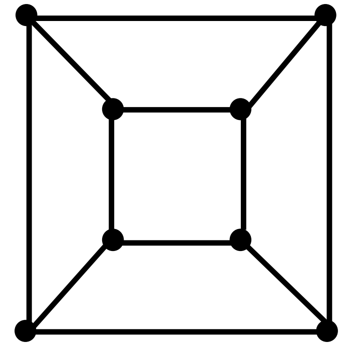
K_4 drawn with no crossings

$\therefore$ K_4 is planar

Example 2: Is Q_3 planar?



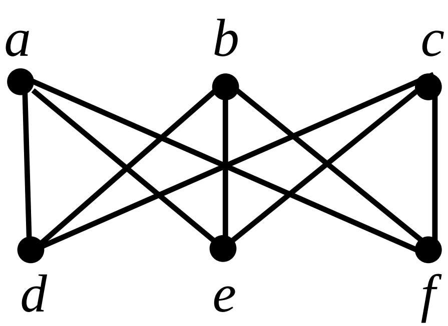
Q_3



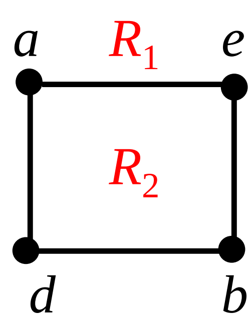
Q_3 drawn with no crossings

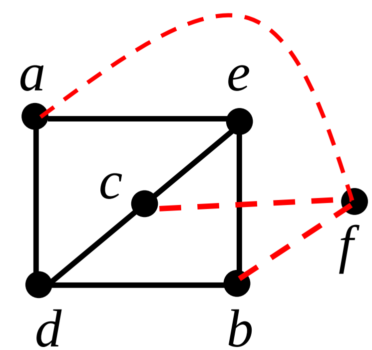
$\therefore$ Q_3 is planar

Example 3: Show that $K_{3,3}$ is nonplanar.



Sol.





任何畫法中，$aebd$都是cycle,
並將平面切成兩個region

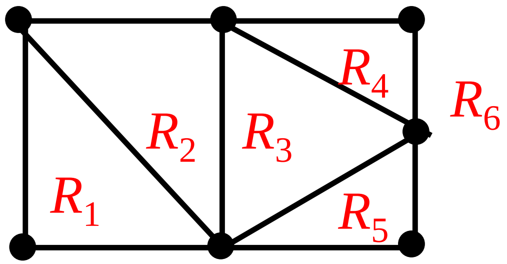
不管 c 在哪個region，都無法
再將 f 放入而使邊不交錯

Exercise: 3, 4

Euler's Formula

A planar representation of a graph splits the plane into regions, including an unbounded region.

Example : How many regions are there in the following graph?



Sol. 6

Thm 1 (Euler's Formula)

Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

Example 4: Suppose that a connected planar graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?

Sol.

$v = 20,\ 2e = 3\times 20 = 60,\ e = 30$

$r = e - v + 2 = 30 - 20 + 2 = 12$

Exercise: 13

Corollary 1

If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

Example 5: Show that K_5 is nonplanar.

Sol.

$v = 5,\ e = 10,$ but $3v - 6 = 9.$

Corollary 2

If G is a connected planar simple graph, then G has a vertex of degree ≤ 5.

pf: Let G be a planar graph of v vertices and e edges.

If $\deg(v) \geq 6$ for every $v \in V(G)$

$$\Rightarrow \sum_{v \in V(G)} \deg(v) \geq 6v$$

$$\Rightarrow 2e \geq 6v \qquad \rightarrow\leftarrow \ (e \leq 3v - 6)$$

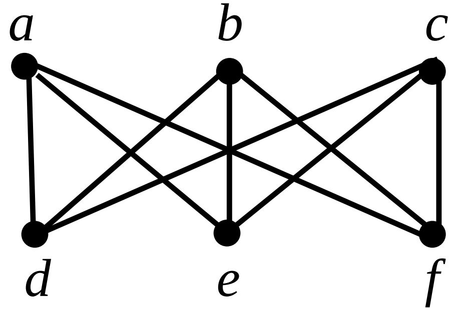
Corollary 3

If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.

Example 6: Show that $K_{3,3}$ is nonplanar by Cor. 3.

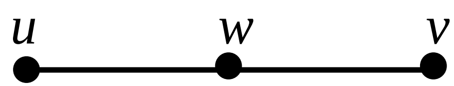
Sol.

Because $K_{3,3}$ has no circuits of length three, and $v = 6$, $e = 9$, but $e = 9 > 2v - 4$.



Kuratowski's Theorem

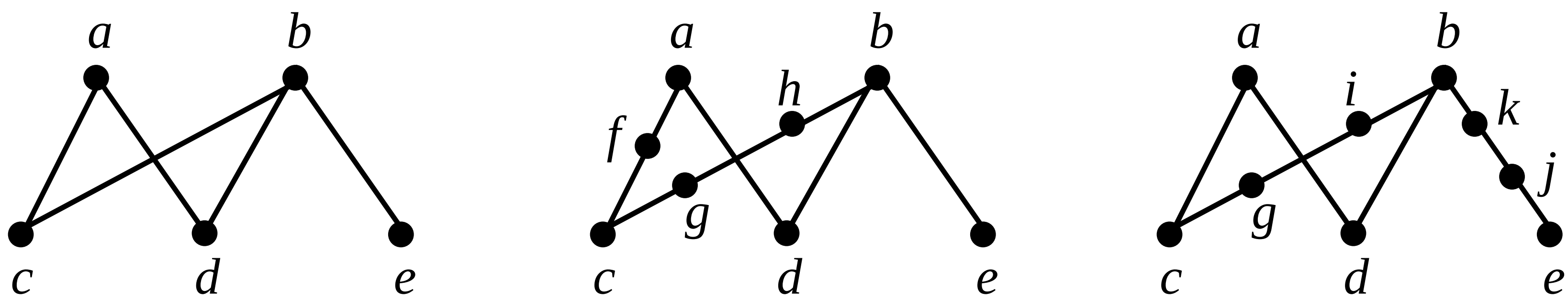
If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{v, w\}$.



Such an operation is called an elementary subdivision.

Two graphs $G_1 = (V_1, E_1)$, $G_2=(V_2, E_2)$ are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivisons.

Example 7: Show that the graphs G_1, G_2, and G_3 are all homeomorphic.



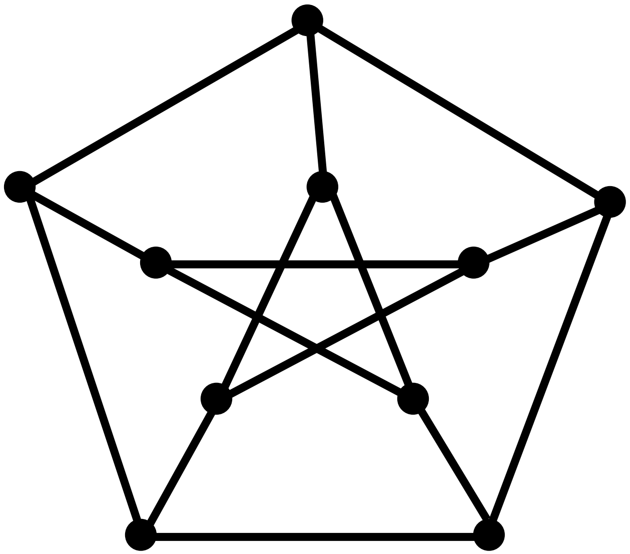
Sol: all three can be obtained from G_1

Exercise: 21

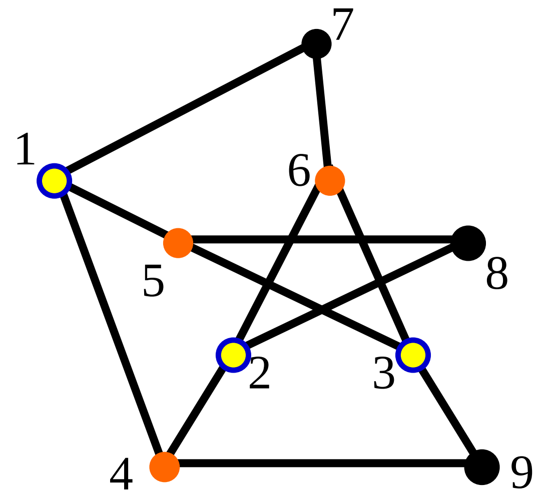
Thm 2. (Kuratowski Theorem)

A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

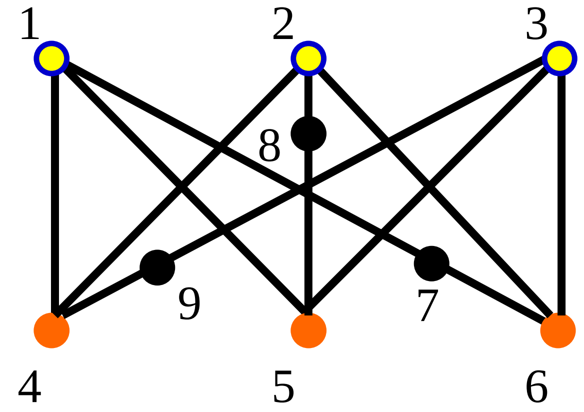
Example 9: Show that the Petersen graph is not planar.



Sol:



It is homeomorphic to $K_{3,3}$.

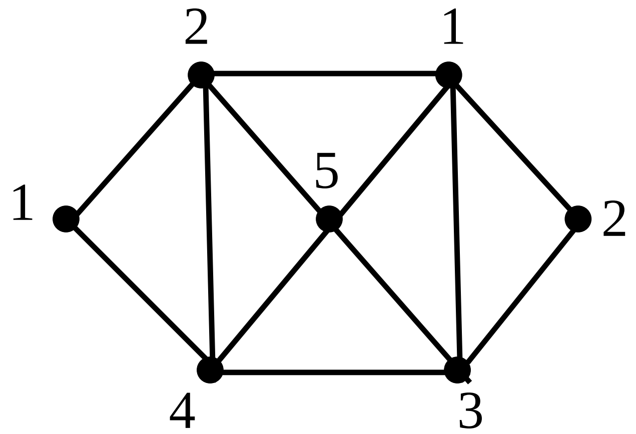


§9.8: Graph Coloring

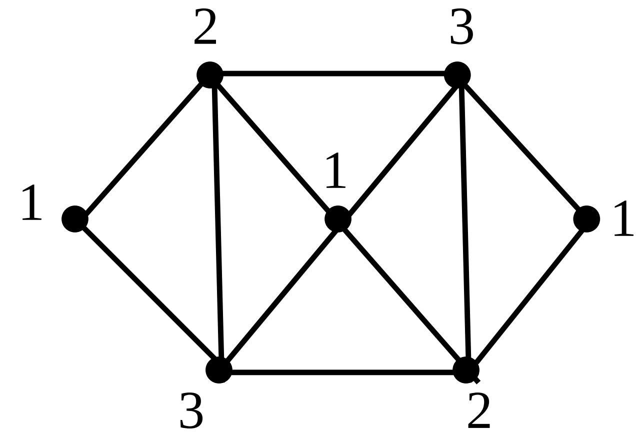
Def. 1:

A *coloring* of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Example:



5-coloring



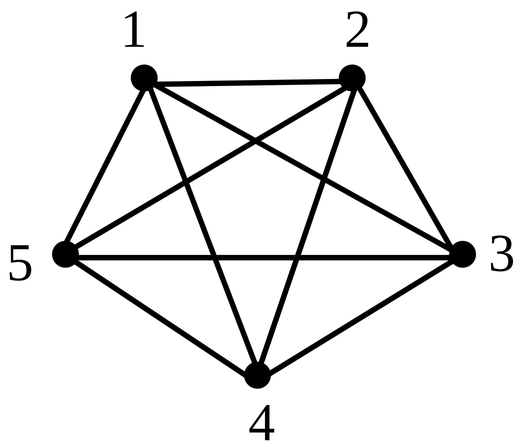
3-coloring

顏色數越少越好

Def. 2:

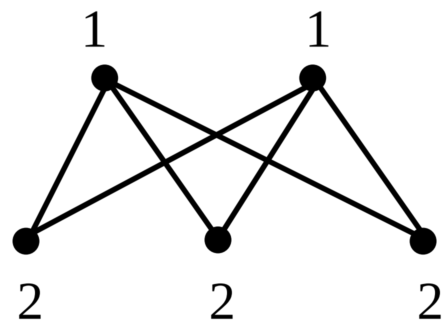
The *chromatic number* of a graph is the least number of colors needed for a coloring of this graph. (denoted by $\chi(G)$)

Example 2: $\chi(K_5)=5$



Note: $\chi(K_n)=n$

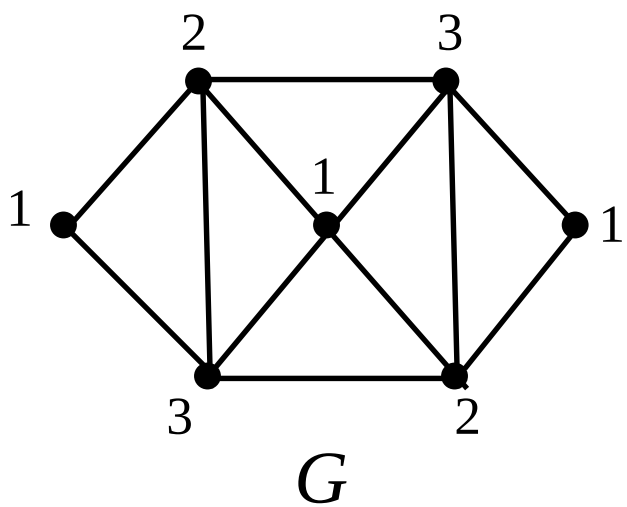
Example: $\chi(K_{2,3}) = 2$.

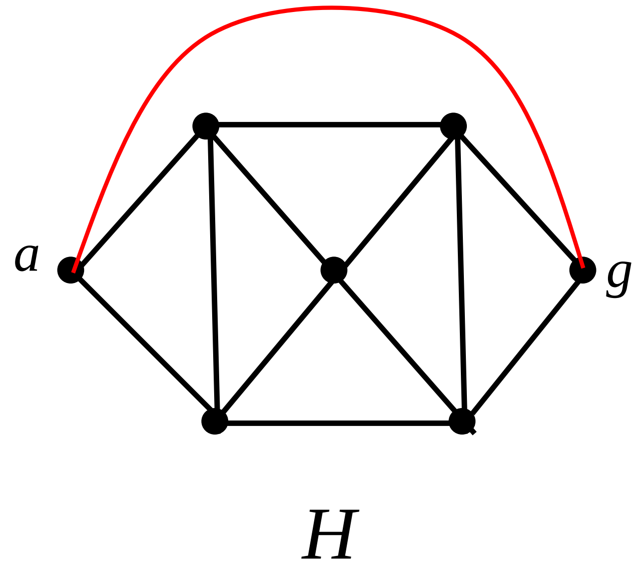


Note: $\chi(K_{m,n}) = 2$

Note: If G is a bipartite graph, $\chi(G) = 2$.

Example 1: What are the chromatic numbers of the graphs G and H?





Sol: G has a 3-cycle

$\Rightarrow \chi(G) \geq 3$

G has a 3-coloring

$\Rightarrow \chi(G) \leq 3$

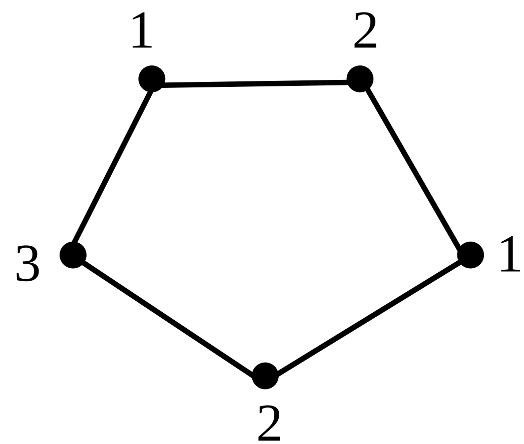
$\Rightarrow \chi(G) = 3$

Sol: any 3-coloring for $H - \{(a,g)\}$ gives the same color to a and g

$\Rightarrow \chi(H) > 3$

4-coloring exists $\Rightarrow \chi(H) = 4$

Example 4: $\chi(C_n) = \begin{cases} 2 \text{ if } n \text{ is even,} \\ 3 \text{ if } n \text{ is odd.} \end{cases}$



C_n is bipartite when n is even.

Thm 1. (The Four Color Theorem)
The chromatic number of a planar graph is no greater than four.

Corollary
Any graph with chromatic number >4 is nonplanar.

Exercise: 8, 9, 15